Intermediate Algebra
for
Science, Technology, Engineering, and Mathematics

Edition ⌈0.99⌉

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World Note icon, Atlas, is from the Noun Project by Delwar Hossain.
W Some world notes are from the website wikipedia.org

Hi! I am Darlene Diaz, a mathematics professor at Santiago Canyon College (SCC) in Orange, California. I am an advocate for Open Educational Resources (OER), where textbooks, notes, workbooks are all free to the students. This means students can access the all materials for the class instantly with no financial burden. For access to online assignments, you can log into MyOpenMath (myopenmath.com) for the e-book, notes, videos, and other materials.

This textbook is an adaptation from Tyler Wallace's *Beginning and Intermediate Algebra*, and my own authored work. I wanted to create a textbook for a STEM-track intermediate algebra, *Intermediate Algebra for Science, Technology, Engineering, and Mathematics*, that was licensed under a Creative Commons Attribution. I used most of Wallace textbook's examples in addition to my authored materials in order to fit the course outline of record for SCC's *Intermediate Algebra* and course identification Math 70X: https://www.c-id.net/resources

The intended audience for this textbook are the students in an Intermediate Algebra course. I want students to be able to read this book with ease and understanding. The language is intended to be more conversational than traditional because I wanted to be able to "talk" to the student. Hence, some language is informal, e.g., the use of plug-n-chug, and should be read with the intent of "talking" to the student. As a professor, I am confident instructors will use a more formal approach, but students need a textbook they can read and easily understand.

Thank you to Tyler Wallace for all his hard work in creating his textbook and materials for us to adapt into our own courses. Without his work, this process would have taken a lot longer and we would have waited longer to offer an OER textbook to our students. Please click the link to view Tyler Wallace's materials: http://wallace.ccfaculty.org/book/book.html.

Thank you to the Academic Senate of California Community Colleges Open Educational Resources Initiative (OERI) for providing the resources and funding to make this project possible. Faculty are so appreciative knowing there is support for these OER projects, and freeing students from financial burden. Thank you to the reviewers that took the time to make suggestions, corrections, and opinions. It is greatly appreciated!

Resources

Here are the resources created for this textbook.

- *Homework* Homework is included in this textbook.
- *Answers to homework* In the end of each chapter, there are answers to select homework exercises.
- *Class Notes* Instructors can download the Class Notes that correspond to this textbook and lecture from the notes on a tablet or projector. Students can print their own copy to take to class every day.
- *MyOpenMath* MyOpenMath (myopenmath.com) is a course management system which includes the online homework assignments, e-book, videos, etc. Feel free to use this site as created by the instructor. Instructors can customize the MOM course from the template.
- *Videos* Videos are included in the MOM template course and also in this textbook. Looking in the margin, for selected examples, there are . Click this icon to view an instructional video of a similar example. A new window opens with the video. No additional software is required to view the videos other than an internet connection.
- Review chapter, Chapter 0: *Arithmetic Essentials*, for a refresher on arithmetic concepts.

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Chapter 0

Arithmetic Essentials

Chapter Objectives

By the end of this chapter, the student should be able to

- Evaluate expressions by applying order of operations, including with absolute value
- Distribute and combine like terms
- Evaluate algebraic expressions
- Translate written statements into algebraic expressions

0.1 Integers

The ability to work comfortably with negative numbers is essential for success in algebra. Hence, we discuss adding, subtracting, multiplying and dividing of integers in this section.

Definition

Integers are zero, all the positive whole numbers and their opposites (negatives).

World Note

The first set of rules for working with negative numbers was written out by the Indian mathematician *Brahmagupta*.

0.1.1 Adding integers

When adding integers we have two cases to consider.

Case 1. *Adding integers with the same signs, i.e., the* ***addends****, the numbers being added, are both positive or both negative.* If the signs are the same, we add the numbers and keep the sign.

Example 0.1

Add: $3 + 6$

Solution.

$$\begin{aligned} 3+6 &\quad \text{Addends are both positive} \to \text{Add } 3+6 \to \text{ Keep the positive} \\ 9 &\quad \text{Sum} \end{aligned}$$

Example 0.2

Add: $-5 + (-3)$

Solution.

$$\begin{aligned} -5+(-3) &\quad \text{Addends are both negative} \to \text{Add } 5+3 \to \text{Keep the negative} \\ -8 &\quad \text{Sum} \end{aligned}$$

Example 0.3

Add: $-7 + (-5)$

Solution.

$$\begin{aligned} -7+(-5) &\quad \text{Addends are both negative} \to \text{Add } 7+5 \to \text{Keep the negative} \\ -12 &\quad \text{Sum} \end{aligned}$$

Case 2. *The signs are different, where one number is positive and one number is negative.* We subtract the absolute values of the numbers and then keep the sign from the larger number. This means if the larger number is positive, the answer is positive, or if the larger number is negative, the answer is negative.

Note

When we say "keep the sign of the larger number," we mean to take the absolute value of each addend, and then determine the larger number, e.g., $-10 + 7$:

$$|-10| = 10 \text{ and } |7| = 7$$

Hence, the larger number is 10 and so we would keep the negative sign in our result.

Example 0.4

Add: $-7+2$

Solution.

$-7+2$	Addends are opposite signs $\rightarrow$ Subtract $7-2$ $\rightarrow$ Keep the sign of the larger number, negative
-5	Sum

Example 0.5

Add: $-4+6$

Solution.

$-4+6$	Addends are opposite signs $\rightarrow$ Subtract $6-4$ $\rightarrow$ Keep the sign of the larger number, positive
2	Sum

Example 0.6

Add: $4+(-3)$

Solution.

$4+(-3)$	Addends are opposite signs $\rightarrow$ Subtract $4-3$ $\rightarrow$ Keep the sign of the larger number, positive
1	Sum

Example 0.7

Add: $7+(-10)$

Solution.

$7+(-10)$	Addends are opposite signs $\rightarrow$ Subtract $10-7$ $\rightarrow$ Keep the sign of the larger number, negative
-3	Sum

0.1.2 Subtracting integers

Subtracting Integers

For subtracting with negative integers, we will rewrite the expression as addition by changing the subtraction sign to an addition sign and rewriting the number after the subtraction sign as its opposite. Then simplify using the methods of adding integers.

This method is often referred to as “adding the opposite.”

Example 0.8

Subtract: $8 - 3$

Solution.

$8 - 3$	Change the sign to addition and rewrite 3 as its opposite
$8 + (-3)$	Addends are opposite signs $\rightarrow$ Subtract $8 - 3 \rightarrow$ Keep the sign of the larger number, positive
5	Difference

Example 0.9

Subtract: $-4 - 6$

Solution.

$-4 - 6$	Change the sign to addition and rewrite 6 as its opposite
$-4 + (-6)$	Addends are same signs $\rightarrow$ Add $4 + 6 \rightarrow$ Keep the sign, negative
-10	Difference

Example 0.10

Subtract: $9 - (-4)$

Solution.

$9 - (-4)$	Change the sign to addition and rewrite 4 as its opposite
$9 + (4)$	Addends are same signs $\rightarrow$ Add $9 + 4 \rightarrow$ Keep the sign, positive
13	Difference

Example 0.11

Subtract: $-6 - (-2)$

Solution.

$-6-(-2)$	Change the sign to addition and rewrite -2 as its opposite
$-6+(2)$	Addends are opposite signs $\rightarrow$ Subtract $6-2 \rightarrow$ Keep the sign of the larger number, negative
-4	Difference

0.1.3 Multiplying and dividing integers

Multiplying and dividing integers

To multiply two integers, we multiply as usual and follow the following properties:

- If the two numbers have signs that are the same, both integers are positive or both are negative, then the product is positive.
- If the two numbers have opposite signs, one number is positive and the other is negative, then the product is negative.

For dividing with integers, we follow the same properties as multiplication.

Example 0.12

Multiply: $(4)(-6)$

Solution.

$(4)(-6)$	Integers have opposite signs $\rightarrow$ Product is negative
-24	Product

Example 0.13

Divide: $\dfrac{-36}{-9}$

Solution.

$\dfrac{-36}{-9}$	Integers are same sign $\rightarrow$ Quotient is positive
4	Quotient

Example 0.14

Multiply: $-2(-6)$

Solution.

$-2(-6)$	Integers are same sign $\rightarrow$ Product is positive
12	Product

Example 0.15

Divide: $\dfrac{15}{-3}$

Solution.

$\dfrac{15}{-3}$	Integers have opposite sign $\rightarrow$ Quotient is negative
-5	Quotient

Warning!

- Be sure to see the difference between problems like $-3 - 8$ and $-3(-8)$.
  - Notice $-3(-8)$ is a multiplication problem because there is nothing between the -3 and the parenthesis. If there is no operation written in between the parts, then we assume that means we are multiplying.
  - The $-3-8$ is a subtraction problem because the subtraction sign separates the -3 from the next number.
- Be sure to distinguish between the patterns for adding and subtracting integers and for multiplying and dividing integers. These operations can look very similar.
  - For example, if the signs match on addition, then we keep the negative, e.g., $-3+(-7) = -10$, but if the signs match on multiplication, then the answer is positive, e.g., $(-3)(-7) = 21$.

0.1.4 Integers Homework

Evaluate each expression.

1. $1-3$

2. $(-6)-(-8)$

3. $(-3)-3$

4. $3-(-5)$

5. $(-7)-(-5)$

6. $3-(-1)$

7. $6-3$

8. $(-5)+3$

9. $2-3$

10. $(-8)-(-5)$

11. $(-2)+(-5)$

12. $5-(-6)$

13. $(-6)+3$

14. $4-7$

15. $(-7)+7$

16. $4-(-1)$

17. $(-6)+8$

18. $(-8)-(-3)$

19. $7-7$

20. $(-4)+(-1)$

21. $(-1)+(-6)$

22. $(-8)+(-1)$

23. $(-1)-8$

24. $5-7$

25. $(-5)+7$

26. $1+(-1)$

27. $8-(-1)$

28. $(-3)+(-1)$

29. $7-3$

30. $(-3)+(-5)$

Find each product.

31. $(4)(-1)$

32. $(10)(-8)$

33. $(-4)(-2)$

34. $(-7)(8)$

35. $(9)(-4)$

36. $(-5)(2)$

37. $(-5)(4)$

38. $(7)(-5)$

39. $(-7)(-2)$

40. $(-6)(-1)$

41. $(6)(-1)$

42. $(-9)(-7)$

43. $(-2)(-2)$

44. $(-3)(-9)$

Find each quotient.

45. $\dfrac{30}{-10}$

46. $\dfrac{-12}{-4}$

47. $\dfrac{30}{6}$

48. $\dfrac{27}{3}$

49. $\dfrac{80}{-8}$

50. $\dfrac{50}{5}$

51. $\dfrac{48}{8}$

52. $\dfrac{54}{-6}$

53. $\dfrac{-49}{-7}$

54. $\dfrac{-2}{-1}$

55. $\dfrac{20}{10}$

56. $\dfrac{-35}{-5}$

57. $\dfrac{-8}{-2}$

58. $\dfrac{-16}{2}$

59. $\dfrac{60}{-10}$

0.2 Fractions

Fractions are a critical part of building a strong algebra foundation. Here, we briefly review reducing, multiplying, dividing, adding, and subtracting fractions.

World Note

The earliest known use of fractions comes from the Middle Kingdom of Egypt around 2000 BC.

0.2.1 Reducing fractions

Reducing fractions

Fractions should always be reduced. We don't always say it, but we know we should do it. We reduce fractions by dividing the numerator and denominator by the same number, called a ***common factor***. We divide by common factors until there are no more common factors between the numerator and denominator.

Example 0.16

Simplify: $\dfrac{36}{84}$

Solution.

$$\begin{aligned} &\frac{36}{84} && \text{Divide by a common factor of 4} \\ \frac{36\div 4}{84\div 4} &= \frac{9}{21} && \text{Divide by a common factor of 3} \\ \frac{9\div 3}{21\div 3} &= \frac{3}{7} && \text{No more common factors} \\ &\frac{3}{7} && \text{Simplified fraction} \end{aligned}$$

Note

In example 0.16, we could have easily reduced the fraction in one step by dividing the numerator and denominator by 12. We also could have simplified in more steps by dividing by 2 twice and then dividing by 3 once (in any order). It is not important which method we use as long as we continue reducing our fraction until there are no common factors between the numerator and denominator.

0.2.2 Multiplying fractions

Multiplying fractions

We multiply fractions by multiplying straight across numerators and denominators:

$$\frac{a}{b} \cdot \frac{c}{d} \implies \frac{a \cdot c}{b \cdot d}$$

Then simplify, if possible.

☠ Be sure to always simplify the fraction! This is a common practice in mathematics and should become habitual after reviewing this section.

Example 0.17

Multiply: $\frac{6}{7} \cdot \frac{3}{5}$

Solution.

$\frac{6}{7} \cdot \frac{3}{5}$	Multiply across numerators and denominators
$\frac{6 \cdot 3}{7 \cdot 5}$	Simplify
$\frac{18}{35}$	No common factors
$\frac{18}{35}$	Product

Note

When multiplying, we can reduce our fractions before or after we multiply. We can either reduce with a single fraction or with several fractions, as long as we use one common factor between the numerator and denominator.

Example 0.18

Multiply: $\frac{25}{24} \cdot \frac{32}{55}$

Solution.

Let's reduce each fraction first, then multiply.

$$\begin{array}{rl}
\dfrac{25}{24}\cdot\dfrac{32}{55} & \text{Reduce 25 \& 55 by a common factor of 5} \\[2ex]
\dfrac{5}{24}\cdot\dfrac{32}{11} & \text{Reduce 24 \& 32 by a common factor of 8} \\[2ex]
\dfrac{5}{3}\cdot\dfrac{4}{11} & \text{Multiply fractions} \\[2ex]
\dfrac{20}{33} & \text{No common factors} \\[2ex]
\dfrac{20}{33} & \text{Product}
\end{array}$$

Example 0.19

Multiply: $\dfrac{25}{24}\cdot\dfrac{32}{55}$

Solution.

Let's multiply first, then reduce the fraction.

$$\begin{array}{rl}
\dfrac{5}{6}\cdot\dfrac{3}{10} & \text{Multiply fractions} \\[2ex]
\dfrac{15}{60} & \text{Reduce by a factor of 15} \\[2ex]
\dfrac{15\div 15}{60\div 15} & \text{Simplify} \\[2ex]
\dfrac{1}{4} & \text{No common factors} \\[2ex]
\dfrac{1}{4} & \text{Product}
\end{array}$$

We can see from examples 0.18 and 0.19 that it doesn't really matter if we first reduce or multiply. As we move further into this course, the student will decide which technique to use for these types of problems.

0.2.3 Dividing fractions

Dividing fractions

Dividing fractions is similar to multiplying fractions with one extra step. We will rewrite the fraction behind the division sign as its reciprocal and change the division sign to multiplication. Then multiply as usual:

$$\frac{a}{b}\div\frac{c}{d}\implies\frac{a}{b}\cdot\frac{d}{c}\implies\frac{a\cdot d}{b\cdot c}$$

Example 0.20

Divide: $\dfrac{21}{16} \div \dfrac{28}{6}$

Solution.

$$\frac{21}{16} \div \frac{28}{6} \quad \text{Rewrite the expression as a product}$$
$$\frac{21}{16} \cdot \frac{6}{28} \quad \text{Reduce the fractions}$$
$$\frac{3}{8} \cdot \frac{3}{4} \quad \text{Multiply fractions}$$
$$\frac{9}{32} \quad \text{Quotient}$$

Sometimes we represent division with fractions by writing a fraction over a fraction, called a **complex fraction**. However, we use the same method, just the presentation changes:

Example 0.21

Divide: $\dfrac{\frac{14}{15}}{\frac{7}{60}}$

Solution.

$$\frac{\frac{14}{15}}{\frac{7}{60}} \quad \text{Rewrite the complex fraction with the division sign}$$
$$\frac{14}{15} \div \frac{7}{60} \quad \text{Rewrite the expression as a product}$$
$$\frac{14}{15} \cdot \frac{60}{7} \quad \text{Reduce the fractions}$$
$$\frac{2}{1} \cdot \frac{4}{1} \quad \text{Multiply fractions}$$
$$\frac{8}{1} \quad \text{Simplify}$$
$$8 \quad \text{Quotient}$$

0.2.4 Adding and subtracting fractions

To add and subtract fractions we will first discuss the least common multiple (LCM). This will lead right into the least common denominator (LCD).

Recall. The *lowest common multiple* (LCM) of a set of factors is the smallest number that is divisible by all factors in the set. If a, b, c are positive integers, then we denote the LCM of this set as $\mathrm{LCM}(a, b, c)$.

Example 0.22

Find LCM$(2, 3, 5)$.

Solution.

We need to think of a multiple of 2, 3, and 5 that is divisible by these numbers. If we multiply 2, 3, and 5, we get

$$2 \cdot 3 \cdot 5 = 30$$

And so, the LCM$(2, 3, 5) = 30$ because 30 is divisible by 2, 3, and 5.

Let's look at a more challenging case:

Example 0.23

Find LCM$(6, 35, 54)$.

Solution.

When the numbers aren't as obvious, then we can use the strategy below to find the LCM:

Step 1. Find the prime factorization of each number in your set.

$$\begin{aligned} 6 &= 2 \cdot 3 \\ 35 &= 5 \cdot 7 \\ 54 &= 2 \cdot 3^3 \end{aligned}$$

Step 2. Look at all the factors and take one of each factor. For the factors with exponents, take the factors with the highest exponent.

$$\begin{array}{ll} 2 & \text{take } 2 \\ 3^3 & \text{take 3 with the highest exponent} \\ 5 & \text{take } 5 \\ 7 & \text{take } 7 \end{array}$$

Step 3. Multiply the numbers found in the previous step. This product is the LCM.

$$\text{LCM}(6, 35, 54) = 2 \cdot 3^3 \cdot 5 \cdot 7 = 1890$$

Definition

The ***lowest common denominator*** (LCD) is the LCM of all denominators given in a set of fractions.

Example 0.24

Find the LCD between $\dfrac{5}{6}$ and $\dfrac{4}{9}$. Rewrite each fraction with the LCD.

Solution.

If we need to obtain the LCD, then we can follow a series of steps.

Step 1. Find the LCD, i.e., the LCM between denominators. In this case, we need to find the

LCM(6, 9).

$$6 = 2 \cdot 3$$
$$9 = 3^2$$

We can see that the $\text{LCM}(6, 9) = 2 \cdot 3^2 = 18$. This is the LCD.

Step 2. Next, we rewrite each fraction with the LCD.

$$\frac{5}{6} \quad \text{Multiply the numerator and denominator by 3}$$
$$\frac{5}{6} \cdot \frac{3}{3} \quad \text{Notice we get 18 in the denominator}$$
$$\frac{15}{18} \quad \text{The denominator is the LCD}\checkmark$$

$$\frac{4}{9} \quad \text{Multiply the numerator and denominator by 2}$$
$$\frac{4}{9} \cdot \frac{2}{2} \quad \text{Notice we get 18 in the denominator}$$
$$\frac{8}{18} \quad \text{The denominator is the LCD}\checkmark$$

Adding and subtracting fractions with the same denominator

When adding and subtracting fractions **with the same denominator**, add and subtract across numerators and keep the denominator the same. Then simplify, if possible.

Example 0.25

Add: $\dfrac{7}{8} + \dfrac{3}{8}$

Solution.

$$\frac{7}{8} + \frac{3}{8} \quad \text{Same denomintaor, add across numerators}$$
$$\frac{10}{8} \quad \text{Reduce by a common factor of 2}$$
$$\frac{5}{4} \quad \text{Sum}$$

Note

We reduce the fraction as the last step. Notice, we add (or subtract) first and bring the fractions together as one fraction, then simplify to lowest terms.

Also, while $\frac{5}{4}$ can be written as the mixed number $1\frac{1}{4}$, in algebra, we hardly use mixed numbers. For this reason we always use improper fractions, not mixed numbers.

Example 0.26

Subtract: $\frac{13}{6} - \frac{9}{6}$

Solution.

$\frac{13}{6} - \frac{9}{6}$	Same denomintaor, subtract across numerators
$\frac{4}{6}$	Reduce by a common factor of 2
$\frac{2}{3}$	Difference

Adding and subtracting fractions with unlike denominators

When adding and subtracting fractions **with unlike denominators**, we rewrite each fraction with the LCD. Then add and subtract as usual.

Example 0.27

Add: $\frac{5}{6} + \frac{4}{9}$

Solution.

$\frac{5}{6} + \frac{4}{9}$	Unlike denominators; $\mathrm{LCD}(6, 9) = 18$
$\frac{5}{6} \cdot \frac{3}{3} + \frac{4}{9} \cdot \frac{2}{2}$	Rewrite each fraction with the LCD
$\frac{15}{18} + \frac{8}{18}$	Same denominator, add across numerators
$\frac{23}{18}$	No common factors
$\frac{23}{18}$	Sum

Example 0.28

Subtract: $\frac{2}{3} - \frac{1}{6}$

Solution.

$$\begin{aligned}
\frac{2}{3} - \frac{1}{6} &\qquad \text{Unlike denominators; LCD}(3,6) = 6\\
\frac{2}{3} \cdot \frac{2}{2} + \frac{1}{6} &\qquad \text{Rewrite each fraction with the LCD}\\
\frac{4}{6} - \frac{1}{6} &\qquad \text{Same denominator, subtract across numerators}\\
\frac{3}{6} &\qquad \text{Reduce by a common factor of 3}\\
\frac{1}{2} &\qquad \text{Difference}
\end{aligned}$$

0.2.5 Fractions Homework

Simplify and leave your answer as an improper fraction.

1. $\frac{42}{12}$
2. $\frac{25}{20}$
3. $\frac{35}{25}$
4. $\frac{24}{9}$
5. $\frac{54}{36}$
6. $\frac{30}{24}$
7. $\frac{36}{27}$
8. $\frac{45}{36}$
9. $\frac{48}{18}$
10. $\frac{27}{18}$
11. $\frac{48}{42}$
12. $\frac{40}{16}$
13. $\frac{16}{12}$
14. $\frac{63}{18}$
15. $\frac{72}{48}$
16. $\frac{80}{60}$
17. $\frac{126}{108}$
18. $\frac{72}{60}$
19. $\frac{160}{140}$
20. $\frac{36}{24}$

Find each product.

21. $(9)\left(\frac{8}{9}\right)$
22. $(2)\left(-\frac{2}{9}\right)$
23. $(-2)\left(\frac{13}{8}\right)$
24. $\left(-\frac{6}{5}\right)\left(-\frac{11}{8}\right)$
25. $(8)\left(\frac{1}{2}\right)$
26. $\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)$
27. $(2)\left(\frac{3}{2}\right)$
28. $\left(\frac{1}{2}\right)\left(-\frac{7}{5}\right)$
29. $(-2)\left(-\frac{5}{6}\right)$
30. $(-2)\left(\frac{1}{3}\right)$
31. $\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)$
32. $\left(-\frac{3}{7}\right)\left(-\frac{11}{8}\right)$
33. $(-2)\left(-\frac{9}{7}\right)$
34. $\left(-\frac{17}{9}\right)\left(-\frac{3}{5}\right)$
35. $\left(\frac{17}{9}\right)\left(-\frac{3}{5}\right)$
36. $\left(\frac{1}{2}\right)\left(\frac{5}{7}\right)$

Find each quotient.

37. $-2 \div \frac{7}{4}$
38. $\frac{-1}{9} \div \frac{-1}{2}$
39. $\frac{-3}{2} \div \frac{13}{7}$
40. $-1 \div \frac{2}{3}$
41. $\frac{8}{9} \div \frac{1}{5}$
42. $\frac{-9}{7} \div \frac{1}{5}$
43. $\frac{-2}{9} \div \frac{-3}{2}$
44. $\frac{1}{10} \div \frac{3}{2}$
45. $\frac{-12}{7} \div \frac{-9}{5}$
46. $-2 \div \frac{-3}{2}$
47. $\frac{5}{3} \div \frac{7}{5}$
48. $\frac{10}{9} \div -6$
49. $\frac{1}{6} \div \frac{-5}{3}$
50. $\frac{-13}{8} \div \frac{-15}{8}$
51. $\frac{-4}{5} \div \frac{-13}{8}$
52. $\frac{5}{3} \div \frac{5}{3}$

Evaluate each expression.

53. $\frac{1}{3}+\left(-\frac{4}{3}\right)$

54. $\frac{3}{7}-\frac{1}{7}$

55. $\frac{11}{6}+\frac{7}{6}$

56. $\frac{3}{5}+\frac{5}{4}$

57. $\frac{2}{5}+\frac{5}{4}$

58. $\frac{9}{8}+\left(-\frac{2}{7}\right)$

59. $1+\left(-\frac{1}{3}\right)$

60. $\left(-\frac{1}{2}\right)+\frac{3}{2}$

61. $\frac{1}{5}+\frac{3}{4}$

62. $\left(-\frac{5}{7}\right)-\frac{15}{8}$

63. $6-\frac{8}{7}$

64. $\frac{3}{2}-\frac{15}{8}$

65. $\left(-\frac{15}{8}\right)+\frac{5}{3}$

66. $(-1)-\left(-\frac{1}{6}\right)$

67. $\frac{5}{3}-\left(-\frac{1}{3}\right)$

68. $\frac{1}{7}+\left(-\frac{11}{7}\right)$

69. $\frac{1}{3}+\frac{5}{3}$

70. $(-2)+\left(-\frac{15}{8}\right)$

71. $(-1)-\frac{2}{3}$

72. $\frac{12}{7}-\frac{9}{7}$

73. $(-2)+\frac{5}{6}$

74. $\frac{1}{2}-\frac{11}{6}$

75. $\frac{11}{8}-\frac{1}{2}$

76. $\frac{6}{5}-\frac{8}{5}$

77. $\left(-\frac{1}{3}\right)+\left(-\frac{8}{5}\right)$

78. $(-6)+\left(-\frac{5}{3}\right)$

79. $(-1)-\left(-\frac{1}{3}\right)$

80. $\frac{3}{2}+\frac{9}{7}$

81. $\left(-\frac{1}{2}\right)-\left(-\frac{3}{5}\right)$

82. $\frac{9}{7}-\left(-\frac{5}{3}\right)$

0.3 Order of operations

When simplifying expressions, it is important that we simplify them in the correct order. Consider the following problem done two different ways:

$$\begin{array}{cc} \underbrace{2+5}\cdot 3 & \text{Add First} \\ \underbrace{7\cdot 3} & \text{Multiply} \\ 21 & \text{Result} \end{array} \qquad\qquad \begin{array}{cc} 2+\underbrace{5\cdot 3} & \text{Multiply} \\ \underbrace{2+15} & \text{Add} \\ 17 & \text{Result} \end{array}$$

The previous example illustrates that if the same problem is done two different ways, we will arrive at two different results. However, only one method is correct. We can think about writing a sentence: it matters where we put punctuation, capital letter, etc. Well, mathematics is very similar and we must follow an order. It turns out the second method, 17, is the correct method. The order of operations ends with the most basic of operations, addition (or subtraction). Before addition is completed, we must do multiplication (or division). Before multiplication is completed, we must do exponents. When we want to do something out of this order, we use grouping symbols, e.g., parenthesis, brackets, absolute value, radical, etc.

Order of operations

We can use the word **PEMDAS** to remember the order of operations, as the first letter of each operation creates the word PEMDAS. Another way to remember the order of operations is to think of a phrase such as "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally," where each word starts with the same letter as the operation. However, it is the author's suggestion to think about PEMDAS as a vertical word written as

P	Parenthesis
E	Exponents
MD	Multiplication & Division
AS	Addition & Subtraction

Note

Multiplying and dividing are done at the same step because they are the same operation (division is just multiplying by the reciprocal). This means multiplication and division must be done in order, left to right. So, some problems we will divide first, others we will multiply first. For adding and subtracting (subtracting is just adding the opposite), we have a similar case.

World Note

The first use of grouping symbols is found in 1646 from the Dutch mathematician's, Franciscus van Schooten's, textbook, *Vieta*. The part of the expression first to be evaluated was represented by a bar. So problems like $2(3+5)$ were written as $2\cdot\overline{3+5}$.

Example 0.29

Simplify the expression completely: $2+3(9-4)^2$

Solution.

$$\begin{aligned}
2+3\underbrace{(9-4)}^2 &\quad \text{Parenthesis}\\
2+3\underbrace{(5)^2} &\quad \text{Exponents}\\
2+\underbrace{3(25)} &\quad \text{Multiply}\\
\underbrace{2+75} &\quad \text{Add}\\
77 &\quad \text{Result}
\end{aligned}$$

Example 0.30

Simplify the expression completely: $30 \div 3 \cdot 2$

Solution.

$$\begin{aligned}
\underbrace{30 \div 3}\cdot 2 &\quad \text{Divide}\\
\underbrace{10 \cdot 2} &\quad \text{Multiply}\\
20 &\quad \text{Result}
\end{aligned}$$

> ***Warning!***
>
> It is very important to remember to multiply and divide from left to right! In example 0.30, if we had multiplied first, we would have obtained 5 as the answer, which is incorrect.

0.3.1 Grouping symbols ( ), { }, [ ]

If there are several parenthesis in a problem, we will start with the inner most parenthesis and work our way out as we apply order of operations to the expression. To avoid confusion with multiple parenthesis, we use different types of grouping symbols such as { } and [ ] and ( ). These grouping symbols all mean the same thing and imply the expression inside must be evaluated first.

Example 0.31

Simplify the expression completely: $2\{8^2 - 7[32 - 4(3^2 + 1)](-1)\}$

Solution.

$$\begin{aligned}
2\{8^2 - 7[32 - 4(\underbrace{3^2}+1)](-1)\} &\quad \text{Innermost parenthesis; exponents}\\
2\{8^2 - 7[32 - 4(\underbrace{9+1})](-1)\} &\quad \text{Add inside those parenthesis}\\
2\{8^2 - 7[32 \underbrace{-4(10)}](-1)\} &\quad \text{Multiply inside inner most parenthesis}
\end{aligned}$$

$$
\begin{array}{rl}
2\{8^2 - 7[\underbrace{32-40}](-1)\} & \text{Subtract inside those parenthesis} \\
2\{\underbrace{8^2}-7[-8](-1)\} & \text{Exponents next} \\
2\{64 \underbrace{-7[-8]}(-1)\} & \text{Multiply left to right , sign with the number} \\
2\{64+\underbrace{56(-1)}\} & \text{Finish multiplying} \\
2\{\underbrace{64-56}\} & \text{Subtract inside parenthesis} \\
\underbrace{2\{8\}} & \text{Multiply} \\
16 & \text{Result}
\end{array}
$$

Example 0.31 illustrates that it can take several steps to complete a problem. The key to successfully simplifying when applying order of operations is to take the time to show your work and do one step at a time. This will reduce the chance of making a mistake along the way.

0.3.2 Grouping symbols– fraction bar

There are several types of grouping symbols that can be used besides parenthesis. One type is a fraction bar. If we have a fraction, the entire numerator and the entire denominator must be evaluated prior to reducing the fraction. In these cases, we can simplify the numerator and denominator simultaneously.

Example 0.32

Simplify the expression completely: $\dfrac{2^4-(-8)\cdot 3}{15\div 5-1}$

Solution.

$$
\begin{array}{rl}
\dfrac{\underbrace{2^4}-(-8)\cdot 3}{\underbrace{15\div 5}-1} & \text{Exponent in the numerator, divide in denominator} \\[2ex]
\dfrac{16-\underbrace{(-8)\cdot 3}}{\underbrace{3-1}} & \text{Multiply in the numerator, subtract in denominator} \\[2ex]
\dfrac{\underbrace{16-(-24)}}{2} & \text{Add the opposite to simplify numerator} \\[2ex]
\dfrac{40}{2} & \text{Reduce} \\[2ex]
20 & \text{Result}
\end{array}
$$

0.3.3 Grouping symbols– absolute value $|\quad|$

Another type of grouping symbol that also has an operation is ***absolute value***. When there is absolute value, we evaluate the expression inside the absolute value first, just as if it were a normal parenthesis. Then take the absolute value.

Recall. The ***absolute value*** of a number is the distance from zero; hence, the absolute value of a number is always positive because distance is always positive. E.g., there's no such thing as running -3 miles, only 3 miles.

Example 0.33

Simplify the expression completely: $1+3|-4^2-(-8)|+2|3+(-5)^2|$

Solution.

$$\begin{aligned}
1+3|-\underbrace{4^2}-(-8)|+2|3+\underbrace{(-5)^2}| &\quad \text{Evaluate absolute values first, exponents}\\
1+3|\underbrace{-16-(-8)}|+2|\underbrace{3+25}| &\quad \text{Add inside absolute values}\\
1+3\underbrace{|-8|}+2\underbrace{|28|} &\quad \text{Evaluate absolute values}\\
1+\underbrace{3(8)}+2(28) &\quad \text{Multiply left to right}\\
1+24+\underbrace{2(28)} &\quad \text{Finish multiplying}\\
\underbrace{1+24}+56 &\quad \text{Add left to right}\\
\underbrace{25+56} &\quad \text{Add}\\
81 &\quad \text{Our Solution}
\end{aligned}$$

☠ ***Warning!***

Example 0.33 illustrates an important point about exponents. Exponents are solely attached to its base number. This means when we see -4^2, only the 4 is squared, giving us $-(4^2)$ or -16, but when the negative is in parentheses, such as $(-5)^2$ the negative is part of the base number and is also squared, giving us a positive solution, 25. Be sure to know the difference to minimize future errors.

0.3.4 Order of Operations Homework

Simplify the expressions completely.

1. $-6 \cdot 4(-1)$

2. $3+(8) \div |4|$

3. $8 \div 4 \cdot 2$

4. $[-9-(2-5)] \div (-6)$

5. $-6+(-3-3)^2 \div |3|$

6. $4-2|3^2-16|$

7. $[-1-(-5)]|3+2|$

8. $\dfrac{2+4|7+2^2|}{4 \cdot 2+5 \cdot 3}$

9. $[6 \cdot 2+2-(-6)]\left(-5+\left|\dfrac{-18}{6}\right|\right)$

10. $\dfrac{-13-2}{2-(-1)^3+(-6)-[-1-(-3)]}$

11. $6 \cdot \dfrac{-8-4+(-4)-[-4-(-3)]}{(4^2+3^2) \div 5}$

12. $\dfrac{2^3+4}{-18-6+(-4)-[-5(-1)(-5)]}$

13. $\dfrac{5+3^2-24 \div 6 \cdot 2}{[5+3(2^2-5)]+|2^2-5|^2}$

14. $(-6 \div 6)^3$

15. $5(-5+6) \cdot 6^2$

16. $7-5+6$

17. $(-2 \cdot 2^3 \cdot 2) \div (-4)$

18. $(-7-5) \div [-2-2-(-6)]$

19. $\dfrac{-10-6}{(-2)^2}-5$

20. $-3-\{3-[-3(2+4)-(-2)]\}$

21. $-4-[2+4(-6)-4-|2^2-5 \cdot 2|]$

22. $2 \cdot (-3)+3-6[-2-(-1-3)]$

23. $\dfrac{-5^2+(-5)^2}{|4^2-2^5|-2 \cdot 3}$

24. $\dfrac{-9 \cdot 2-(3-6)}{1-(-2+1)-(-3)}$

25. $\dfrac{13+(-3)^2+4(-3)+1-[-10-(-6)]}{\{[4+5] \div [4^2-3^2(4-3)-8]\}+12}$

0.4 Properties of algebra

In algebra, we will often need to simplify an expression. There are three basic forms of simplifying which we will discuss in this section.

World Note

The term "Algebra" comes from the Arabic word *al-jabr* which means "reunion." It was first used in Iraq in 830 AD by Mohammad ibn-Musa al-Khwarizmi.

Definition

An algebraic expression consists of coefficients, variables, and terms. Given an algebraic expression, a

- ***coefficient*** is the number in front of the variable.
- ***variable*** is a letter representing any number.
- ***term*** is a product of a coefficient and variable(s).

For example,

$$t \qquad 2x \qquad 3st \qquad 7x^2 \qquad 5ab^3c$$

are all examples of terms because each is a product of a coefficient and variable(s).

0.4.1 Evaluating expressions

The first form of simplifying expressions is evaluating expressions. Given particular values for each variable, we can simplify the expression by replacing the variables with its corresponding values.

Example 0.34

Evaluate $p(q+6)$ when $p=3$ and $q=5$.

Solution.

$$\begin{aligned} p(q+6) &\quad \text{Replace } p \text{ with 3 and } q \text{ with 5} \\ (3)((5)+6) &\quad \text{Evaluate parenthesis} \\ (3)(11) &\quad \text{Multiply} \\ 33 &\quad \text{Result} \end{aligned}$$

Note

Whenever we replace a variable, we will put the new number inside a set of parenthesis. Notice the 3 and 5 in example 0.34 are in parenthesis. This is to preserve operations that are sometimes lost in a simple replacement. Sometimes the parenthesis won't make a difference, but it is a good habit to always use them to prevent potential future arithmetic errors.

Example 0.35

Evaluate $x + zx(3-z)\left(\frac{x}{3}\right)$ when $x=-6$ and $z=-2$.

Solution.

$$\begin{aligned} x + zx(3 - z)\left(\frac{x}{3}\right) & \quad \text{Replace } x \text{ with } 6 \text{ and } z \text{ with } 2 \\ (-6) + (-2)(-6)(3 - (-2))\left(\frac{(-6)}{3}\right) & \quad \text{Evaluate parenthesis} \\ -6 + (5)(-2) & \quad \text{Multiply left to right} \\ -6 + 12(5)(-2) & \quad \text{Multiply left to right} \\ -6 + 60(-2) & \quad \text{Multiply} \\ -6 - 120 & \quad \text{Subtract} \\ -126 & \quad \text{Result} \end{aligned}$$

0.4.2 Like terms

It is common in the study of Algebra that the values of the variables are unknown. In this case, we simplify by combining *like terms*.

Definition

Two terms are ***like terms*** if the base variable(s) and exponent on each variable are identical.

For example, $3x^2y$ and $-7x^2y$ are like terms because they both contain the same base variables, x and y, and the exponents on x (the x is squared on both terms) and y are the same.

Combining like terms

If two terms are *like terms*, we add (or subtract) the ***coefficients***, then keep the variables (and exponents on the corresponding variable) the same.

Example 0.36

Simplify: $5x - 2y - 8x + 7y$

Solution.

$$\begin{aligned} 5x - 2y - 8x + 7y & \quad \text{Combine like terms } 5x - 8x \text{ and } -2y + 7y \\ -3x + 5y & \quad \text{Result} \end{aligned}$$

Example 0.37

Simplify: $8x^2 - 3x + 7 - 2x^2 + 4x - 3$

Solution.

$8x^2 - 3x + 7 - 2x^2 + 4x - 3$	Combine like terms $8x^2 - 2x^2$ and $-3x + 4x$ and $7 - 3$
$6x^2 + x + 4$	Result

Note

As we combine like terms, we interpret subtraction signs as part of the following term. Hence, if we see a subtraction sign, we treat the following term as a negative term.

Notice, when we write the simplified result, it is common practice to write the expression in **standard form**, terms written with descending exponents. E.g., looking at the result in example 0.37, we wrote $6x^2 + x + 4$, where the x^2 term is written first since it is the largest exponent and then the x term. We always write the term with just the coefficient at the end, e.g., 4.

0.4.3 Distribution

The final method for simplifying algebraic expressions is *distribution*. Many times we are given algebraic expressions with sets of parenthesis and terms directly in front of the expressions (as product). By using the *distributive property*, we can rewrite the expression without parenthesis.

Distributive property

Property. The ***distributive property*** is a product between one term and a sum or difference of two or more terms:

$$a(b + d) = a \cdot b + a \cdot d$$

Example 0.38

Simplify: $4(2x - 7)$

Solution.

$4(2x - 7)$	Multiply each term by 4
$4 \cdot 2x - 4 \cdot 7$	Simplify
$8x - 28$	Result

Example 0.39

Simplify: $-7(5x - 6)$

Solution.

$$\begin{aligned} -7(5x-6) &\quad \text{Multiply each term by } -7 \\ (-7)\cdot 5x-(-7)\cdot 6 &\quad \text{Simplify} \\ -35x+42 &\quad \text{Result} \end{aligned}$$

Warning!

In the previous example, we use the fact that the sign is attached with the number, i.e., we treat the -6 as a negative number: $(-7)(-6) = 42$, a positive number. The most common error in using the distributive property is a sign (negatives) error. Be very careful with your signs!

It is possible to distribute a negative through parenthesis. When there is a negative in front of parenthesis, we can think of the negative as a -1. We don't always write it, but we know it's there. Then we distribute the -1 as usual.

Example 0.40

Simplify $-(4x-5y+6)$

Solution.

$$\begin{aligned} -(4x-5y+6) &\quad \text{Negative can be thought of as } -1 \\ -1(4x-5y+6) &\quad \text{Multiply each term by } -1 \\ (-1)4x-(-1)5y+(-1)6 &\quad \text{Simplify} \\ -4x+5y-6 &\quad \text{Result} \end{aligned}$$

0.4.4 Putting it all together

Distributing through parenthesis and combining like terms can be combined into one problem. Order of operations implies multiplication (distribute) first, then add or subtract (combine like terms). Thus, we first distribute and then combine like terms.

Example 0.41

Simplify: $5+3(2x-4)$

Solution.

$$\begin{aligned} 5+3(2x-4) &\quad \text{Distribute} \\ 5+6x-12 &\quad \text{Combine like terms} \\ -7+6x &\quad \text{Rewrite in standard form} \\ 6x-7 &\quad \text{Result} \end{aligned}$$

Example 0.42

Simplify: $3x-2(4x-5)$

Solution.

$$\begin{aligned} 3x - 2(4x - 5) & \quad \text{Distribute} \\ 3x - 8x + 10 & \quad \text{Combine like terms} \\ -5x + 10 & \quad \text{Result} \end{aligned}$$

Example 0.43

Simplify: $2(5x - 8) - 6(4x + 3)$

Solution.

$$\begin{aligned} 2(5x - 8) - 6(4x + 3) & \quad \text{Distribute} \\ 10x - 16 - 24x - 18 & \quad \text{Combine like terms} \\ -14x - 34 & \quad \text{Result} \end{aligned}$$

Example 0.44

Simplify: $4(3x - 8) - (2x - 7)$

Solution.

$$\begin{aligned} 4(3x - 8) - (2x - 7) & \quad \text{Treat the negative as a } -1 \\ 4(3x - 8) - 1(2x - 7) & \quad \text{Distribute} \\ 12x - 32 - 2x + 7 & \quad \text{Combine like terms} \\ 10x - 25 & \quad \text{Result} \end{aligned}$$

0.4.5 Properties of Algebra Homework

Evaluate each expression given the values for each variable.

1. $p+1+q-m;\quad m=1, p=3, q=4$

2. $p-\frac{pq}{6};\quad p=6 \text{ and } q=5$

3. $c^2-(a-1);\quad a=3 \text{ and } c=5$

4. $5j+\frac{kh}{2};\quad h=5, j=4, k=2$

5. $\frac{4-(p-m)}{2}+q;\quad m=4, p=6, q=6$

6. $m+n+m+\frac{n}{2};\quad m=1 \text{ and } n=2$

7. $q-p-(q-1-3);\quad p=3, q=6$

8. $y^2+y-z;\quad y=5, z=1$

9. $\frac{6+z-y}{3};\quad y=1, z=4$

10. $x+6z-4y;\quad x=6, y=4, z=4$

11. $5(b+a)+1+c;\quad a=2, b=6, c=5$

12. $z+x-(1^2)^3;\quad x=5, z=4$

13. $3+z-1+y-1;\quad y=5, z=4$

14. $p+(q-r)(6-p);\quad p=6, q=5, r=5$

15. $y-[4-y-(z-x)];\quad x=3, y=1, z=6$

16. $4z-\big(x+x-(z-z)\big);\quad x=3, z=2$

17. $k\times 3^2-(j+k)-5;\quad j=4, k=5$

18. $zx-\left(z-\frac{4+x}{6}\right);\quad x=2, z=6$

19. $a^3(c^2-c);\quad a=3, c=2$

20. $5+qp+pq-q;\quad p=6, q=3$

Simplify.

21. $r-9+10$

22. $n+n$

23. $8v+7v$

24. $-7x-2x$

25. $k-2+7$

26. $x-10-6x+1$

27. $m-2m$

28. $9n-1+n+4$

29. $-4x+2-4$

30. $4b+6+1+7b$

31. $-x+8x$

32. $-7a-6+5$

33. $-8p+5p$

34. $1-10n-10$

35. $1-r-6$

36. $-4b+9b$

37. $-8(x-4)$

38. $8n(n+9)$

39. $7k(-k+6)$

40. $-6(1+6x)$

41. $8m(5-m)$

42. $-9x(4-x)$

43. $-9b(b-10)$

44. $-8n(5+10n)$

45. $3(8v+9)$

46. $-(-5+9a)$

47. $10x(1+2x)$

48. $-2(n+1)$

49. $-2p(9p-1)$

50. $4(8n-2)$

51. $-4(1+7r)$

52. $2x(8x-10)$

53. $9(b+10)+5b$

54. $-3x(1-4x)-4x^2$

55. $-4k^2-8k(8k+1)$

56. $1-7(5+7p)$

57. $-10-4(n-5)$

58. $4(x+7)+8(x+4)$

59. $-8(n+6)-8n(n+8)$

60. $7(7+3v)+10(3-10v)$

61. $2n(-10n+5)-7(6-10n)$

62. $5(1-6k)+10(k-8)$

63. $(8n^2-3n)-(5+4n^2)$

64. $(5p-6)+(1-p)$

65. $(2-4v^2)+(3v^2+2v)$

66. $(4-2k^2)+(8-2k^2)$

67. $(x^2-8)+(2x^2-7)$

68. $4v-7(1-8v)$

69. $-8x + 9(-9x + 9)$

70. $-9 - 10(1 + 9a)$

71. $-10(x - 2) - 3$

72. $-6(5 - m) + 3m$

73. $-2r(1 + 4r) + 8r(-r + 4)$

74. $9(6b + 5) - 4b(b + 3)$

75. $-7(4x - 6) + 2(10x - 10)$

76. $-3(4 + a) + 6a(9a + 10)$

77. $-7(4x + 3) - 10(10x + 10)$

78. $(7x^2 - 3) - (5x^2 + 6x)$

79. $(3x^2 - x) - (7 - 8x)$

80. $(2b - 8) + (b - 7b^2)$

81. $(7a^2 + 7a) - (6a^2 + 4a)$

82. $(3 - 7n^2) + (6n^2 + 3)$

0.5 Arithmetic Essentials: Answers to the Homework Exercises

Integers

1. -2
3. -6
5. -2
7. 3
9. -1
11. -7
13. -3
15. 0
17. 2
19. 0
21. -7
23. -9
25. 2
27. 9
29. 4
31. -4
33. 8
35. -36
37. -20
39. 14
41. -6
43. 4
45. -3
47. 5
49. -10
51. 6
53. 7
55. 2
57. 4
59. -6

Fractions

1. $\frac{7}{2}$
3. $\frac{7}{5}$
5. $\frac{3}{2}$
7. $\frac{4}{3}$
9. $\frac{8}{3}$
11. $\frac{8}{7}$
13. $\frac{4}{3}$
15. $\frac{3}{2}$
17. $\frac{7}{6}$
19. $\frac{8}{7}$
21. 8
23. $-\frac{13}{4}$
25. 4
27. 3
29. $\frac{5}{3}$
31. $\frac{3}{4}$
33. $\frac{18}{7}$
35. $-\frac{17}{15}$
37. $-\frac{8}{7}$
39. $-\frac{21}{26}$
41. $\frac{40}{9}$
43. $\frac{4}{27}$
45. $\frac{20}{21}$
47. $\frac{25}{21}$
49. $-\frac{1}{10}$
51. $\frac{32}{65}$
53. -1
55. 3
57. $\frac{33}{20}$
59. $\frac{2}{3}$
61. $\frac{19}{20}$
63. $\frac{34}{7}$
65. $-\frac{5}{24}$
67. 2
69. 2
71. $-\frac{5}{3}$
73. $-\frac{7}{6}$
75. $\frac{7}{8}$
77. $-\frac{29}{15}$
79. $-\frac{2}{3}$
81. $\frac{1}{10}$

Order of Operations

1. 24
3. 4
5. 6
7. 20
9. -40
11. -18
13. 2
15. 180
17. 8
19. -9
21. 28
23. 0
25. 3

Properties of Algebra

1. 7
3. 23
5. 7
7. 1
9. 36
11. 29
13. 14
15. 8
17. 6
19. 54
21. $r+1$
23. $15v$
25. $k+5$
27. $-m$
29. $-8x+32$
31. $-7k^2+42k$
33. $40m-8m^2$
35. $-9b^2+90b$
37. $14b+90$
39. $-68k^2-8k$

41. $10 - 4n$

43. $-72n - 48 - 8n^2$

45. $-20n^2 + 80n - 42$

47. $4n^2 - 3n - 5$

49. $-v^2 + 2v + 2$

51. $3x^2 - 15$

53. $11b + 7$

55. $-7a - 1$

57. $-9 - 10n$

59. $5b$

61. $5 - 9a$

63. $-2n - 2$

65. $32n - 8$

67. $16x^2 - 20x$

69. $-89x + 81$

71. $-10x + 17$

73. $30r - 16r^2$

75. $-8x + 22$

77. $-128x - 121$

79. $3x^2 + 7x - 7$

81. $a^2 + 3a$

Chapter 1

Linear Equations

Chapter Objectives

By the end of this chapter, the student should be able to

- Solve linear equations (simple, dual-side variables, infinitely many solutions or no solution, rational coefficients)
- Solve literal equations with several variables for one of the variables
- Solve absolute value equations
- Apply solving techniques to application problems including perimeter, mark-up/discount, triangles, and interest problems

1.1 Linear equations

Definition

A ***linear equation*** is an equation where the highest exponent on the given variables is one. A *linear equation in one variable* is an equation with one variable with exponent one, e.g.,

$$ax + b = c,$$

where a is called the *coefficient* of x, and b and c are *constant coefficients*.

Solving linear equations is an important and fundamental skill in algebra. In algebra, we are often presented with a problem where the answer is unknown. This is usually represented by a variable. There will be times when there are multiple unknowns and we use algebra techniques to solve for the variable.

Warning!

Be sure to understand the difference between an expression and an equation.

Recall. We simplify expressions and solve equations. Hence, expressions do not contain an equal sign, =, and we only distribute and combine like terms. Equations contain an equal sign, =, and we solve for the variable in addition to distributing and combining like terms.

The result to an expression is an expression and the result to an equation is a number (with the exception of when the result is infinitely many solutions or no solution).

1.1.1 Verifying solutions

Example 1.1

Is $x = -5$ a solution to the equation $4x + 16 = -4$?

Solution.

We substitute $x = -5$ into the equation and determine if the left side is equal to the right side.

$$\begin{aligned} 4(-5) + 16 &\stackrel{?}{=} -4 && \text{Multiply } 4(-5) \\ -20 + 16 &\stackrel{?}{=} -4 && \text{Add } -20 + 16 \\ -4 &= -4 && \checkmark \text{ True} \end{aligned}$$

Hence, $x = -5$ is a solution to the equation $4x + 16 = -4$.

Example 1.2

Is $x = 3$ a solution to the equation $4x + 16 = -4$?

Solution.

We substitute $x = 3$ into the equation and determine if the left side is equal to the right side.

$$\begin{aligned} 4(3) + 16 &\stackrel{?}{=} -4 && \text{Multiply } 4(3) \\ 12 + 16 &\stackrel{?}{=} -4 && \text{Add } 12 + 16 \\ 28 &\neq -4 && \text{✗ False} \end{aligned}$$

Hence, $x = 3$ is not a solution to the equation $4x + 16 = -4$.

✎ **Note**

Example 1.2 reiterates that the solution to each equation is unique. Once we see that at one solution makes the equation true, then we look no further. The exception is when we have an identity, which we discuss later in this chapter.

1.1.2 One-step equations

Depending on the complexity of the problem, this "guess and check" method is not very efficient. Thus, we take a more algebraic approach for solving equations. Here we will focus on equations that only require one step to solve. While these equations often seem very fundamental, it is important to master the pattern for solving these problems so we can solve more complex problems.

Addition property of equations

Given an equation, $a = b$, the following is an equivalent statement:

$$a + c = b + c$$

i.e., we can add any number to one side of the equation as long as we add the same number to the other side. Often, we use this property to isolate the variable.

Example 1.3

Solve for x: $x + 7 = -5$

Solution.

$$\begin{aligned} x + 7 &= -5 && \text{Isolate } y \text{ by adding the opposite of 7} \\ x + 7 + (-7) &= -5 + (-7) && \text{Simplify} \\ x &= -12 && \text{Solution} \end{aligned}$$

Recall, it is encouraged for the student to check the obtained answer by verifying the solution:

$$\begin{aligned} (-12) + 7 &\stackrel{?}{=} -5 \\ -5 &= -5 && \checkmark \text{ True} \end{aligned}$$

Example 1.4

Solve for y: $4 + y = 8$

Solution.

$$\begin{aligned} 4 + y &= 8 && \text{Isolate } y \text{ by adding the opposite of 4} \\ 4 + y + (-4) &= 8 + (-4) && \text{Simplify} \\ y &= 4 && \text{Solution} \end{aligned}$$

Example 1.5

Solve for y: $7 = y + 9$

Solution.

$$\begin{aligned} 7 &= y + 9 && \text{Isolate } y \text{ by adding the opposite of 9} \\ 7 + (-9) &= y + 9 + (-9) && \text{Simplify} \\ -2 &= y && \text{Rewrite with } y \text{ on the left side} \\ y &= -2 && \text{Solution} \end{aligned}$$

Example 1.6

Solve for x: $5 = 8 + x$

Solution.

$$\begin{aligned} 5 &= 8 + x && \text{Isolate } x \text{ by adding the opposite of } 8 \\ 5 + (-8) &= 8 + x + (-8) && \text{Simplify} \\ -3 &= x && \text{Rewrite with } x \text{ on the left side} \\ x &= -3 && \text{Solution} \end{aligned}$$

☠ *Warning!*

Be sure to check your answer by verifying the solution! It only take a few seconds and will save you time and future common errors in the long run. ☺

Example 1.7

Solve for y: $y - 5 = 4$

Solution.

$$\begin{aligned} y - 5 &= 4 && \text{Isolate } y \text{ by adding the opposite of } -5 \\ y - 5 + (5) &= 4 + (5) && \text{Simplify} \\ y &= 9 && \text{Solution} \end{aligned}$$

Example 1.8

Solve for y: $-10 = y - 7$

Solution.

$$\begin{aligned} -10 &= y - 7 && \text{Isolate } y \text{ by adding the opposite of } -7 \\ -10 + (7) &= y - 7 + (7) && \text{Simplify} \\ y &= -3 && \text{Solution} \end{aligned}$$

Multiplication property of equations

Given an equation, $a = b$, the following is an equivalent statement:

$$ac = bc,$$

where $c \neq 0$, i.e., we can multiply both sides of the equation by a nonzero number. Often, we use this property to isolate the variable when there is a coefficient in front of the variable.

Example 1.9

Solve for y: $4y = 20$

Solution.

$$\begin{aligned} 4y &= 20 && \text{Isolate } y \text{ by multiplying by the reciprocal of } 4\\ \frac{1}{4}\cdot 4y &= 20\cdot\frac{1}{4} && \text{Simplify}\\ y &= 5 && \text{Solution} \end{aligned}$$

Note

In example 1.9, we multiplied by the ***reciprocal***. The product of a number and its reciprocal is one, i.e., if c is a number, then its reciprocal is $\frac{1}{c}$ because

$$c\cdot\frac{1}{c} = 1$$

Example 1.10

Solve for y: $-5y = 30$

Solution.

$$\begin{aligned} -5y &= 30 && \text{Isolate } y \text{ by multiplying by the reciprocal of } -5\\ -\frac{1}{5}\cdot -5y &= 30\cdot -\frac{1}{5} && \text{Simplify}\\ y &= -6 && \text{Solution} \end{aligned}$$

Example 1.11

Solve for x: $-42 = -7x$

Solution.

$$\begin{aligned} -42 &= -7x && \text{Isolate } x \text{ by multiplying by the reciprocal of } -7\\ -\frac{1}{7}\cdot -42 &= -7x\cdot -\frac{1}{7} && \text{Simplify}\\ 6 &= x && \text{Rewrite with } x \text{ on the left side}\\ x &= 6 && \text{Solution} \end{aligned}$$

Example 1.12

Solve for x: $\frac{x}{-7} = -2$

Solution.

$$\begin{aligned} \frac{x}{-7} &= -2 && \text{Isolate } x \text{ by multiplying by the reciprocal of } -\frac{1}{7} \\ -7 \cdot \frac{x}{-7} &= -2 \cdot -7 && \text{Simplify} \\ x &= 14 && \text{Solution} \end{aligned}$$

Example 1.13

Solve for x: $\frac{x}{5} = -3$

Solution.

$$\begin{aligned} \frac{x}{5} &= -3 && \text{Isolate } x \text{ by multiplying by the reciprocal of } \frac{1}{5} \\ 5 \cdot \frac{x}{5} &= -3 \cdot 5 && \text{Simplify} \\ x &= -15 && \text{Solution} \end{aligned}$$

The processes described above is fundamental for solving equations. Once these processes are mastered, we are given problems that involve multiple steps. These problems may seem more complex, but the processes and patterns are the same.

World Note

The study of algebra was originally called "*Cossic Art*" from the Latin, meaning the study of "things," which we now call variables.

1.1.3 Two-step equations

After mastering the techniques for solving one-step equations, we are ready to consider two-step equations. As we solve two-step equations, the important thing to remember is that everything works in reverse. When working with one-step equations, we learned that in order to clear a "plus five" in the equation, we would subtract five (or add its opposite) from both sides. We learned that to clear "divided by seven," we multiply by seven (or multiply by its reciprocal) on both sides. When solving for our variable, we use order of operations in reverse. This means we will apply the addition property first, then the multiplication property second (then exponents, and, finally, any parenthesis or grouping symbols, but that's another lesson).

Example 1.14

Solve for y: $4y - 20 = -8$

Solution.

We have one extra term on the same side as the variable term, $4y$. We will first isolate the variable term, then isolate the variable by applying the properties of equations:

$$\begin{aligned} 4y - 20 &= -8 && \text{Add the opposite of } -20 \text{ to each side} \\ 4y - 20 + 20 &= -8 + 20 && \text{Simplify} \\ 4y &= 12 && \text{Multiply by the reciprocal of 4} \\ \frac{1}{4} \cdot 4y &= 12 \cdot \frac{1}{4} && \text{Simplify} \\ y &= 3 && \text{Solution} \end{aligned}$$

Let's verify the solution:

$$\begin{aligned} 4(3) - 20 &\stackrel{?}{=} -8 \\ 12 - 20 &\stackrel{?}{=} -8 \\ -8 &= -8 \qquad \checkmark \text{ True} \end{aligned}$$

Thus, $y = 3$ is the solution to the equation.

The same process is used to solve any two-step equations. Add or subtract first, then multiply or divide. Recall, the method never changes, just problems do.

Example 1.15

Solve for w: $5w + 7 = 7$

Solution.

$$\begin{aligned} 5w + 7 &= 7 && \text{Add the opposite of 7 to each side} \\ 5w + 7 + (-7) &= 7 + (-7) && \text{Simplify} \\ 5w &= 0 && \text{Multiply by the reciprocal of 5} \\ \frac{1}{5} \cdot 5w &= 0 \cdot \frac{1}{5} && \text{Simplify} \\ w &= 0 && \text{Solution} \end{aligned}$$

Note

Notice in example 1.15 the seven canceled out completely. Many students get stuck at this point. Let's not forget that the product of a number and zero is zero. Hence, zero is the solution to the equation.

Warning!

A common error students make with two-step equations is with negative signs. Remember, the sign always stays with the coefficient.

Example 1.16

Solve for t: $4 - 2t = 10$

Solution.

$$\begin{aligned}
4-2t &= 10 && \text{Add the opposite of 4 to each side}\\
4-2t+(-4) &= 10+(-4) && \text{Simplify}\\
-2t &= 6 && \text{Multiply by the reciprocal of } -2\\
-\frac{1}{2}\cdot -2t &= 6\cdot -\frac{1}{2} && \text{Simplify}\\
t &= -3 && \text{Solution}
\end{aligned}$$

Example 1.17

Solve for n: $8-n=2$

Solution.

$$\begin{aligned}
8-n &= 2 && \text{Add the opposite of 8 to each side}\\
8-n+(-8) &= 2+(-8) && \text{Simplify}\\
-n &= -6 && \text{Rewrite } -n \text{ as } -1n\\
-1n &= -6 && \text{Multiply by the reciprocal of } -1\\
-1\cdot -1n &= -6\cdot -1 && \text{Simplify}\\
n &= 6 && \text{Solution}
\end{aligned}$$

Example 1.18

Solve for y: $-3y+7=-8$

Solution.

$$\begin{aligned}
-3y+7 &= -8 && \text{Add the opposite of 7 to each side}\\
-3y+7+(-7) &= -8+(-7) && \text{Simplify}\\
-3y &= -15 && \text{Multiply by the reciprocal of } -3\\
-\frac{1}{3}\cdot -3y &= -15\cdot -\frac{1}{3} && \text{Simplify}\\
y &= 5 && \text{Solution}
\end{aligned}$$

Example 1.19

Solve for x: $-3=\dfrac{x}{5}-4$

Solution.

Notice the variable term is on the right side of the equation. However, we still follow the method as usual:

$$\begin{aligned}
-3 &= \frac{x}{5} - 4 && \text{Add the opposite of } -4 \text{ to each side}\\
-3 + 4 &= \frac{x}{5} - 4 + 4 && \text{Simplify}\\
1 &= \frac{x}{5} && \text{Multiply by the reciprocal of } \frac{1}{5}\\
5 \cdot 1 &= \frac{x}{5} \cdot 5 && \text{Simplify}\\
5 &= x && \text{Rewrite the solution with the variable on the left side}\\
x &= 5 && \text{Solution}
\end{aligned}$$

As problems in algebra become more complex, the method remains the same. In fact, as we solve problems like those in the next example, each one of them will have several steps to solve, but the last two steps are a two-step equation. This is a critical reason to master two-step equations.

Example 1.20

$$3x^2 + 4 = y + 6 \qquad \frac{1}{x-8} + \frac{1}{x} = \frac{1}{3} \qquad \sqrt{5y-5} + 1 = y \qquad \log_5(2y-4) = 1$$

World Note

Persian mathematician *Omar Khayyam* would solve algebraic problems geometrically by intersecting graphs rather than solving them algebraically.

1.1.4 General equations

Often as we are solving linear equations, we will need to do some preliminary work so that it is more familiar for us to solve. This section will focus on manipulating an equation in such a way that we can use our method for solving two-step equations to obtain the solution.

One such issue that needs to be addressed is parentheses. Sometimes parenthesis can get in the way of solving an equation. As you might expect, we can rewrite an equation without parenthesis by using the distributive property.

Distributive property of equations

The following is an equivalent statement:

$$c(a+b) = ca + cb$$

where $c \neq 0$, i.e., we can multiply each term in the parenthesis by a nonzero number.

Example 1.21

Solve for y: $4(2y-6) = 16$

Solution.

Notice the first step is distributing, then it is solved like any other two-step equation.

$$\begin{aligned} 4(2y-6) &= 16 && \text{Distribute} \\ 8y-24 &= 16 && \text{Add the opposite of } -24 \text{ to each side} \\ 8y-24+24 &= 16+24 && \text{Simplify} \\ 8y &= 40 && \text{Multiply by the reciprocal of 8} \\ \frac{1}{8}\cdot 8y &= 40\cdot\frac{1}{8} && \text{Simplify} \\ y &= 5 && \text{Solution} \end{aligned}$$

Example 1.22

Solve for p: $3(2p-4)+9=15$

Solution.

Notice the first step is distributing, then it is solved like any other two-step equation.

$$\begin{aligned} 3(2p-4)+9 &= 15 && \text{Distribute} \\ 6p-12+9 &= 15 && \text{Combine like terms} \\ 6p-3 &= 15 && \text{Add the opposite of } -3 \text{ to each side} \\ 6p-3+3 &= 15+3 && \text{Simplify} \\ 6p &= 18 && \text{Multiply by the reciprocal of 6} \\ \frac{1}{6}\cdot 6p &= 18\cdot\frac{1}{6} && \text{Simplify} \\ p &= 3 && \text{Solution} \end{aligned}$$

Example 1.23

Solve for a: $3(4a-5)=4(2a+1)+5$

Solution.

Notice we have variable terms on each side of the equation. We will not only distribute first, but also isolate the variable term before applying the two-step method:

$$\begin{aligned} 3(4a-5) &= 4(2a+1)+5 && \text{Distribute} \\ 12a-15 &= 8a+4+5 && \text{Combine like terms} \\ 12a-15 &= 8a+9 && \text{Isolate the variable term by adding the opposite of } 8a \\ & && \text{to each side} \\ 12a-15+(-8a) &= 8a+9+(-8a) && \text{Simplify} \\ 4a-15 &= 9 && \text{Add the opposite of } -15 \text{ to each side} \\ 4a-15+15 &= 9+15 && \text{Simplify} \\ 4a &= 24 && \text{Multiply by the reciprocal of 4} \\ \frac{1}{4}\cdot 4a &= 24\cdot\frac{1}{4} && \text{Simplify} \\ a &= 6 && \text{Solution} \end{aligned}$$

A general strategy to solving linear equations

In general, there is a 5-step process to solving any linear equation. While all five steps aren't always needed, this can serve as a guide for solving equations.

Step 1. Apply the distributive property to rewrite the equation without parenthesis.

Step 2. Combine like terms on each side of the equation.

Step 3. Isolate the variable term by applying the addition property.

Step 4. Solve the equation by applying the multiplication property.

Step 5. ✓Verify the solution.

World Note

The Chinese developed a method for solving equations that involved finding each digit one at a time about 2,000 years ago.

There are two special cases when solving linear equations. The first is illustrated in the next two examples. Notice we start by distributing and moving the variables all to the same side.

Example 1.24

Solve for y: $3(2y-5) = 6y - 15$

Solution.

$$\begin{aligned} 3(2y-5) &= 6y-15 && \text{Distribute} \\ 6y-15 &= 6y-15 && \text{Isolate the variable term by adding the opposite of } 6y \\ & && \text{to each side} \\ 6y-15+(-6y) &= 6y-15+(-6y) && \text{Simplify} \\ -15 &\stackrel{?}{=} -15 && \text{Is this true? Yes } \checkmark \\ -15 &= -15 \end{aligned}$$

Notice all variables canceled and we are left with only a statement with numbers. In this case, the statement we are left with is a true statement, i.e., -15 does equal -15, and so there are infinitely many solutions to this equation. In this special case, when we obtain infinitely many solutions, then the solution is *all real numbers*. When the solution is *all real numbers*, we call this equation an *identity*.

Example 1.25

Solve for n: $2(3n-5) - 4n = 2n + 7$

Solution.

$$2(3n-5) - 4n = 2n + 7 \qquad \text{Distribute}$$

$$
\begin{aligned}
6n - 10 - 4n &= 2n + 7 && \text{Combine like terms} \\
2n - 10 &= 2n + 7 && \text{Isolate the variable term by adding the opposite of } 2n \\
& && \text{to each side} \\
-10 &\stackrel{?}{=} 7 && \text{Is this true? No ✗} \\
-10 &\neq 7
\end{aligned}
$$

Notice all variables canceled and we are left with only a statement with numbers. In this case, the statement we are left with is a false statement, i.e., -10 does not equal 7, and so there is no solution to this equation. In this special case, when we obtain a false statement, then the solution is *no solution* and we call this equation a *contradiction*.

Conditional, identity and contradiction equations

There are three types of equations we saw in the examples and in examples 1.24 and 1.25:

- An equation is a ***conditional equation*** when there is *one unique solution*, i.e., $x =$ number.
- An equation is an ***identity*** when we obtain *infinitely many solutions*, as in example 1.24, where the solution to the equation is *all real numbers*. Hence, we can choose any number for the variable and this number will make the statement true.
- An equation is a ***contradiction*** when the solution is *no solution*, as in example 1.25. Hence, there is no number for the variable such that the statement is true.

1.1.5 Solving equations with fractions

Often when solving linear equations, we work with an equation with fraction coefficients. We can solve these problems as usual. Recall, the methods never change, just problems.

Example 1.26

Solve for y: $\frac{3}{4}y - \frac{7}{2} = \frac{5}{6}$

Solution.

$$
\begin{aligned}
\frac{3}{4}y - \frac{7}{2} &= \frac{5}{6} && \text{Isolate the variable term by adding the opposite of } -\frac{7}{2} \\
\frac{3}{4}y - \frac{7}{2} + \frac{7}{2} &= \frac{5}{6} + \frac{7}{2} && \text{Simplify}
\end{aligned}
$$

Notice, in order to add $\frac{5}{6} + \frac{7}{2}$, we need to obtain the LCD. Since the LCD$(2, 6) = 6$, we can rewrite the right side with the LCD:

$$\frac{3}{4}y = \frac{5}{6} + \frac{21}{6} \quad \text{Combine like terms}$$

$$\frac{3}{4}y = \frac{26}{6} \quad \text{Simplify } \frac{26}{6}$$

$$\frac{3}{4}y = \frac{13}{3} \quad \text{Solve by multiplying by the reciprocal of } \frac{3}{4}$$

$$\frac{4}{3} \cdot \frac{3}{4}y = \frac{13}{3} \cdot \frac{4}{3} \quad \text{Simplify}$$

$$y = \frac{52}{9} \quad \text{Solution}$$

While this process does help us arrive at the correct solution, the fractions can make the process quite difficult and we are more inclined to make errors. Hence, we have an alternate method called *clearing denominators*.

Clearing denominators

We can easily clear denominators in an equation by multiplying each term by the LCD. After completing this step, the fractions are cleared and we can work with a more familiar type of equation.

Let's try example 1.26 again, but, now, by clearing denominators first, then solving.

Example 1.27

Solve for y: $\frac{3}{4}y - \frac{7}{2} = \frac{5}{6}$

Solution.

$$\frac{3}{4}y - \frac{7}{2} = \frac{5}{6} \quad \text{Multiply each term by the LCD}(2,4,6) = 12$$

$$12 \cdot \frac{3}{4}y - 12 \cdot \frac{7}{2} = 12 \cdot \frac{5}{6} \quad \text{Simplify}$$

$$\frac{36}{4}y - \frac{84}{2} = \frac{60}{6} \quad \text{Reduce the fractions}$$

$$9y - 42 = 10 \quad \text{Add the opposite of } -42 \text{ to each side}$$

$$9y - 42 + 42 = 10 + 42 \quad \text{Simplify}$$

$$9y = 52 \quad \text{Multiply by the reciprocal of 9}$$

$$\frac{1}{9} \cdot 9y = 52 \cdot \frac{1}{9} \quad \text{Simplify}$$

$$y = \frac{52}{9} \quad \text{Solution}$$

Thus, the solution is $y = \frac{52}{9}$ and example 1.26 is a *conditional equation*. Also, we observe that as soon as we multiplied each term by the LCD, we cleared the denominators and the equation no longer contained fractions.

Example 1.28

Solve for t: $\frac{2}{3}t - 2 = \frac{3}{2}t + \frac{1}{6}$

Solution.

$$\frac{2}{3}t - 2 = \frac{3}{2}t + \frac{1}{6} \quad \text{Multiply each term by the LCD}(2,3,6) = 6$$

$$6 \cdot \frac{2}{3}t - 6 \cdot 2 = 6 \cdot \frac{3}{2}t + 6 \cdot \frac{1}{6} \quad \text{Simplify}$$

$$\frac{12}{3}t - 12 = \frac{18}{2}t + \frac{6}{6} \quad \text{Reduce the fractions}$$

$$4t - 12 = 9t + 1 \quad \text{Isolate the variable term by adding the opposite of } 9t \text{ to each side}$$

$$4t - 12 + (-9t) = 9t + 1 + (-9t) \quad \text{Simplify}$$

$$-5t - 12 = 1 \quad \text{Add the opposite of } -12 \text{ to each side}$$

$$-5t - 12 + 12 = 1 + 12 \quad \text{Simplify}$$

$$-5t = 13 \quad \text{Multiply by the reciprocal of } -5$$

$$-\frac{1}{5} \cdot -5t = 13 \cdot -\frac{1}{5} \quad \text{Simplify}$$

$$t = -\frac{13}{5} \quad \text{Solution}$$

Thus, the solution is $t = -\frac{13}{5}$ and example 1.28 is a *conditional equation*.

Note

In example 1.28, we could write the solution as -2.6 given we were using a calculator. A good rule of thumb is if you start with fractions, the solution should also be a fraction. Hence, since there were fractions in original problem, we will leave the solution as a fraction.

World Note

The Egyptians were among the first to study fractions and linear equations. The most famous mathematical document from Ancient Egypt is the *Rhind Papyrus*, where the unknown variable was called "heap."

1.1.6 Solving equations with distributing fractions

We can use this same method if parenthesis are in the given problem. We will first distribute the coefficient in front of the parenthesis, then clear denominators.

Example 1.29

Solve for y: $\frac{3}{2}\left(\frac{5}{9}y + \frac{4}{27}\right) = 3$

Solution.

$$
\begin{aligned}
\frac{3}{2}\left(\frac{5}{9}y+\frac{4}{27}\right) &= 3 && \text{Distribute } \frac{3}{2} \text{ and reduce}\\
\frac{5}{6}y+\frac{2}{9} &= 3 && \text{Multiply each term by the LCD}(6,9)=18\\
18\cdot\frac{5}{6}y+18\cdot\frac{2}{9} &= 18\cdot 3 && \text{Multiply and simplify}\\
15y+4 &= 54 && \text{Add the opposite of 4 to each side}\\
15y+4+(-4) &= 54+(-4) && \text{Simplify}\\
15y &= 50 && \text{Multiply by the reciprocal of 15}\\
\frac{1}{15}\cdot 15y &= 50\cdot\frac{1}{15} && \text{Simplify}\\
y &= \frac{50}{15} && \text{Reduce}\\
y &= \frac{10}{3} && \text{Solution}
\end{aligned}
$$

Thus, the solution is $y=\frac{10}{3}$ and example 1.29 is a *conditional equation.*

Example 1.30

Solve for q: $\frac{1}{4}q-\frac{1}{2}=\frac{1}{3}\left(\frac{3}{4}q+6\right)-\frac{7}{2}$

Solution.

$$
\begin{aligned}
\frac{1}{4}q-\frac{1}{2} &= \frac{1}{3}\left(\frac{3}{4}q+6\right)-\frac{7}{2} && \text{Distribute } \frac{1}{3} \text{ and reduce}\\
\frac{1}{4}q-\frac{1}{2} &= \frac{1}{4}q+2-\frac{7}{2} && \text{Multiply each term by the LCD}(2,4)=4\\
4\cdot\frac{1}{4}q-4\cdot\frac{1}{2} &= 4\cdot\frac{1}{4}q+4\cdot 2-4\cdot\frac{7}{2} && \text{Multiply and reduce}\\
q-2 &= q+8-14 && \text{Combine like terms}\\
q-2 &= q-6 && \text{Isolate the variable term by adding the opposite of } q\\
& && \text{to each side}\\
q-2+(-q) &= q-6+(-q) && \text{Simplify}\\
-2 &\stackrel{?}{=} -6 && \text{Is this true? No ✗}\\
-2 &\neq -6 && \text{This implies there is no solution}
\end{aligned}
$$

Since we obtain a false statement, there is no solution and this equation is called a *contradiction.*

1.1.7 Linear Equations Homework

Solve the one-step equations.

1. $v+9=16$

2. $x-11=-16$

3. $30=a+20$

4. $x-7=-26$

5. $13=n-5$

6. $340=-17x$

7. $-9=\dfrac{n}{12}$

8. $20v=-160$

9. $340=20n$

10. $16x=320$

11. $-16+n=-13$

12. $p-8=-21$

13. $180=12x$

14. $20b=-200$

15. $\dfrac{r}{14}=\dfrac{5}{14}$

16. $-7=a+4$

17. $10=x-4$

18. $13a=-143$

19. $\dfrac{p}{20}=-12$

20. $9+m=-7$

21. $14=b+3$

22. $-14=x-18$

23. $-1+k=5$

24. $-13+p=-19$

25. $22=16+m$

26. $4r=-28$

27. $\dfrac{5}{9}=\dfrac{b}{9}$

28. $-20x=-80$

29. $\dfrac{1}{2}=\dfrac{a}{8}$

30. $\dfrac{k}{13}=-16$

31. $21=x+5$

32. $m-4=-13$

33. $3n=24$

34. $-17=\dfrac{x}{12}$

35. $n+8=10$

36. $v-16=-30$

37. $-15=x-16$

38. $-8k=120$

39. $-15=\dfrac{x}{9}$

40. $-19=\dfrac{n}{20}$

Solve the two-step equations.

41. $5+\dfrac{n}{4}=4$

42. $102=-7r+4$

43. $-8n+3=-77$

44. $0=-6v$

45. $-8=\dfrac{x}{5}-6$

46. $0=-7+\dfrac{k}{2}$

47. $-12+3x=0$

48. $24=2n-8$

49. $2=-12+2r$

50. $\dfrac{b}{3}+7=10$

51. $152=8n+64$

52. $-16=8a+64$

53. $56+8k=64$

54. $-2x+4=22$

55. $-20=4p+4$

56. $-5=3+\dfrac{n}{2}$

57. $\dfrac{r}{8}-6=-5$

58. $-40=4n-32$

59. $87=3-7v$

60. $-x+1=-11$

61. $-2=-2m+12$

62. $27=21-3x$

63. $-4-b=8$

64. $-2+\dfrac{x}{2}=4$

65. $-5=\dfrac{a}{4}-1$

66. $-6=15+3p$

67. $-5m+2=27$

68. $-37 = 8 + 3x$

69. $-8 + \frac{n}{12} = -7$

70. $\frac{x}{1} - 8 = -8$

71. $-11 = -8 + \frac{v}{2}$

72. $-2x - 3 = -29$

73. $-4 - 3n = -16$

74. $67 = 5m - 8$

75. $9 = 8 + \frac{x}{6}$

76. $\frac{m}{4} - 1 = -2$

77. $-80 = 4x - 28$

78. $33 = 3b + 3$

79. $3x - 3 = -3$

80. $4 + \frac{a}{3} = 1$

Solve.

81. $2 - (-3a - 8) = 1$

82. $-5(-4 + 2v) = -50$

83. $66 = 6(6 + 5x)$

84. $0 = -8(p - 5)$

85. $-2 + 2(8x - 7) = -16$

86. $-21x + 12 = -6 - 3x$

87. $-1 - 7m = -8m + 7$

88. $1 - 12r = 29 - 8r$

89. $20 - 7b = -12b + 30$

90. $-32 - 24v = 34 - 2v$

91. $-2 - 5(2 - 4m) = 33 + 5m$

92. $-4n + 11 = 2(1 - 8n) + 3n$

93. $-6v - 29 = -4v - 5(v + 1)$

94. $2(4x - 4) = -20 - 4x$

95. $-a - 5(8a - 1) = 39 - 7a$

96. $-57 = -(-p + 1) + 2(6 + 8p)$

97. $-2(m - 2) + 7(m - 8) = -67$

98. $50 = 8(7 + 7r) - (4r + 6)$

99. $-8(n - 7) + 3(3n - 3) = 41$

100. $-61 = -5(5r - 4) + 4(3r - 4)$

101. $-2(8n - 4) = 8(1 - n)$

102. $-3(-7v + 3) + 8v = 5v - 4(1 - 6v)$

103. $-7(x - 2) = -4 - 6(x - 1)$

104. $-6(8k + 4) = -8(6k + 3) - 2$

105. $-2(1 - 7p) = 8(p - 7)$

106. $2(-3n + 8) = -20$

107. $2 - 8(-4 + 3x) = 34$

108. $32 = 2 - 5(-4n + 6)$

109. $-55 = 8 + 7(k - 5)$

110. $-(3 - 5n) = 12$

111. $-3n - 27 = -27 - 3n$

112. $56p - 48 = 6p + 2$

113. $4 + 3x = -12x + 4$

114. $-16n + 12 = 39 - 7n$

115. $17 - 2x = 35 - 8x$

116. $-25 - 7x = 6(2x - 1)$

117. $-7(1 + b) = -5 - 5b$

118. $-8(8r - 2) = 3r + 16$

119. $-8n - 19 = -2(8n - 3) + 3n$

120. $-4 + 4k = 4(8k - 8)$

121. $16 = -5(1 - 6x) + 3(6x + 7)$

122. $7 = 4(n - 7) + 5(7n + 7)$

123. $-8(6 + 6x) + 4(-3 + 6x) = -12$

124. $-76 = 5(1 + 3b) + 3(3b - 3)$

125. $-6(x - 8) - 4(x - 2) = -4$

126. $-4(1 + a) = 2a - 8(5 + 3a)$

127. $-6(x - 3) + 5 = -2 - 5(x - 5)$

128. $-(n + 8) + n = -8n + 2(4n - 4)$

129. $-5(x + 7) = 4(-8x - 2)$

130. $8(-8n + 4) = 4(-7n + 8)$

Solve.

131. $\frac{3}{5}(1+p)=\frac{21}{20}$

132. $0=-\frac{5}{4}\left(x-\frac{6}{5}\right)$

133. $\frac{3}{4}-\frac{5}{4}m=\frac{113}{24}$

134. $\frac{635}{72}=-\frac{5}{2}\left(-\frac{11}{4}+x\right)$

135. $2b+\frac{9}{5}=-\frac{11}{5}$

136. $\frac{3}{2}\left(\frac{7}{3}n+1\right)=\frac{3}{2}$

137. $-a-\frac{5}{4}\left(-\frac{8}{3}a+1\right)=-\frac{19}{4}$

138. $\frac{55}{6}=-\frac{5}{2}\left(\frac{3}{2}p-\frac{5}{3}\right)$

139. $\frac{16}{9}=-\frac{4}{3}\left(-\frac{4}{3}n-\frac{4}{3}\right)$

140. $-\frac{5}{8}=\frac{5}{4}\left(r-\frac{3}{2}\right)$

141. $-\frac{11}{3}+\frac{3}{2}b=\frac{5}{2}\left(b-\frac{5}{3}\right)$

142. $-\left(-\frac{5}{2}x-\frac{3}{2}\right)=-\frac{3}{2}+x$

143. $\frac{45}{16}+\frac{3}{2}n=\frac{7}{4}n-\frac{19}{16}$

144. $\frac{3}{2}\left(v+\frac{3}{2}\right)=-\frac{7}{4}v-\frac{19}{6}$

145. $\frac{47}{9}+\frac{3}{2}x=\frac{5}{3}\left(\frac{5}{2}x+1\right)$

146. $-\frac{1}{2}=\frac{3}{2}k+\frac{3}{2}$

147. $\frac{3}{2}n-\frac{8}{3}=-\frac{29}{12}$

148. $\frac{11}{4}+\frac{3}{4}r=\frac{163}{32}$

149. $-\frac{16}{9}=-\frac{4}{3}\left(\frac{5}{3}+n\right)$

150. $\frac{3}{2}-\frac{7}{4}v=-\frac{9}{8}$

151. $\frac{41}{9}=\frac{5}{2}\left(x+\frac{2}{3}\right)-\frac{1}{3}x$

152. $\frac{1}{3}\left(-\frac{7}{4}k+1\right)-\frac{10}{3}k=-\frac{13}{8}$

153. $-\frac{1}{2}\left(\frac{2}{3}x-\frac{3}{4}\right)-\frac{7}{2}x=-\frac{83}{24}$

154. $\frac{2}{3}\left(m+\frac{9}{4}\right)-\frac{10}{3}=-\frac{53}{18}$

155. $\frac{1}{12}=\frac{4}{3}x+\frac{5}{3}\left(x-\frac{7}{4}\right)$

156. $\frac{7}{6}-\frac{4}{3}n=-\frac{3}{2}n+2\left(n+\frac{3}{2}\right)$

157. $-\frac{149}{16}-\frac{11}{3}r=-\frac{7}{4}r-\frac{5}{4}\left(-\frac{4}{3}r+1\right)$

158. $-\frac{7}{2}\left(\frac{5}{3}a+\frac{1}{3}\right)=\frac{11}{4}a+\frac{25}{8}$

159. $-\frac{8}{3}-\frac{1}{2}x=-\frac{4}{3}x-\frac{2}{3}\left(-\frac{13}{4}x+1\right)$

160. $\frac{1}{3}n+\frac{29}{6}=2\left(\frac{4}{3}n+\frac{2}{3}\right)$

1.2 Absolute value equations

When solving equations with absolute value, the solution may result in more than one possible answer because, recall, absolute value is just distance from zero. Since the integer -4 has distance 4 units from zero, and 4 has distance 4 units from zero, then there are *two* integers that have distance 4 from zero, $-4, 4$. We extend this concept to algebraic absolute value equations. This is illustrated in the following example.

Example 1.31

Solve for x: $|x| = 7$

Solution.

$$\begin{aligned} |x| &= 7 && \text{Expression in the absolute value can be positive or negative} \\ x = 7 \text{ or } x &= -7 && \text{Solution} \end{aligned}$$

Let's think about the solution set. The equation is asking for all numbers in which the distance from zero is 7. Well, there are two integers that have a distance 7 from zero, -7 and 7. Hence, the solution set $\{-7, 7\}$.

World Note

The first set of rules for working with negative numbers came from 7^{th} century India. However, in 1758, more than a thousand years later, British mathematician Francis Maseres claimed that negatives "Darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple."

Definition

Absolute value for linear equations in one variable is given by

$$\text{If } |x| = a, \text{ then } x = a \text{ or } x = -a$$

where a is a real number.

When we have an equation with absolute value, it is important to first isolate the absolute value, then remove the absolute value by applying the definition.

Example 1.32

Solve for x: $5 + |x| = 8$

Solution.

$$\begin{aligned} 5 + |x| &= 8 && \text{Isolate the absolute value by subtracting 5 from each side} \\ |x| &= 3 && \text{Rewrite as two linear equations} \\ x = 3 \text{ or } x &= -3 && \text{Solution} \end{aligned}$$

Thus, the solution set is $\{-3, 3\}$.

Example 1.33

Solve for x: $-4|x| = -20$

Solution.

$$\begin{aligned} -4|x| &= -20 && \text{Isolate the absolute value} \\ \frac{-4|x|}{-4} &= \frac{-20}{-4} && \text{Divide each side by } -4 \\ |x| &= 5 && \text{Rewrite as two linear equations} \\ x = 5 \text{ or } x &= -5 && \text{Solution} \end{aligned}$$

Thus, the solution set is $\{-5, 5\}$.

Warning!

Never combine the inside of the absolute value with factors or terms from outside the absolute value. We always have to isolate the absolute value first, then apply the definition to obtain two equations without the absolute value.

Example 1.34

Solve for y: $5|y| - 4 = 26$

Solution.

$$\begin{aligned} 5|y| - 4 &= 26 && \text{Isolate the absolute value term by adding 4 to each side} \\ 5|y| &= 30 && \text{Divide each side by 5} \\ |y| &= 6 && \text{Rewrite as two linear equations} \\ y = 6 \quad \text{or} \quad y &= -6 && \text{Solution} \end{aligned}$$

Thus, the solution set is $\{-6, 6\}$.

1.2.1 Absolute value equations with different solutions

Often, we will have linear arguments inside the absolute value which changes the solution. Previously, all solution sets have been opposite integers, but in these cases, the solution sets contain different sized numbers.

Example 1.35

Solve for t: $|2t - 1| = 7$

Solution.

$$\begin{aligned} |2t - 1| &= 7 && \text{The absolute value term is isolated. Rewrite as two linear equations.} \\ 2t - 1 = 7 \quad \text{or} \quad 2t - 1 &= -7 && \text{Solve each equation.} \end{aligned}$$

Notice we have two equations to solve where each equation results in a different solution. In any case, we solve as usual.

$$\begin{array}{lcl} 2t-1=7 & & 2t-1=-7 \\ 2t=8 & \text{or} & 2t=-6 \\ t=4 & & t=-3 \end{array}$$

Thus, the solution set is $\{-3, 4\}$.

1.2.2 Multiple-step absolute value equations

Example 1.36

Solve for x: $2-4|2x+3|=-18$

Solution.

To isolate the absolute value, we first apply the addition rule for equations. Then apply the multiplication rule for equations.

$$\begin{array}{rl} 2-4|2x+3|=-18 & \text{Isolate the absolute value term by subtracting 2 from each side} \\ -4|2x+3|=-20 & \text{Divide each side by } -4 \\ |2x+3|=5 & \text{Rewrite as two linear equations.} \\ 2x+3=5 \quad \text{or} \quad 2x+3=-5 & \end{array}$$

Solve each equation.

$$\begin{array}{lcl} 2x+3=5 & & 2x+3=-5 \\ 2x=2 & \text{or} & 2x=-8 \\ x=1 & & x=-4 \end{array}$$

We now have obtained two solutions, $x=1$ and $x=-4$. Thus, the solution set is $\{-4, 1\}$

1.2.3 Equations with two absolute values

In this case, we have an absolute value on each side of the equals sign. However, even though there are two absolute values, we apply the same process. Recall, methods never change, only problems.

Example 1.37

Solve for m: $|2m-7|=|4m+6|$

Solution.

In order to apply the definition, we rewrite this equation as two linear equations, but with the left side as its positive and negative value:

$$\begin{array}{rl} |2m-7|=|4m+6| & \text{Rewrite as two linear equations.} \\ 2m-7=4m+6 \quad \text{or} \quad 2m-7=-(4m+6) & \end{array}$$

Now, we can solve as usual. Be sure to distribute the negative for the equation on the right.

$$\begin{aligned}2m-7&=4m+6\\-13&=2m\\-\frac{13}{2}&=m\end{aligned}\qquad\text{or}\qquad\begin{aligned}2m-7&=-(4m+6)\\2m-7&=-4m-6\\6m-7&=-6\\6m&=1\\m&=\frac{1}{6}\end{aligned}$$

This gives two solutions, $m=-\frac{13}{2}$ or $m=\frac{1}{6}$. Thus, the solution set is $\left\{-\frac{13}{2},\frac{1}{6}\right\}$.

Note

In Example 1.37, because there are absolute value expressions on both sides of the equation, we could have easily applied the definition to the left side and obtained

$$2m-7=4m-6\qquad\text{or}\qquad-(2m-7)=4m-6$$

Then solved each linear equation as usual and obtained the same results.

1.2.4 Special cases

As we are solving absolute value equations, it is important to be aware of special cases. Remember, the result after evaluating absolute value must always be non-negative.

Example 1.38

Solve for x: $7+|2x-5|=4$

Solution.

$7+\lvert 2x-5\rvert=4$	Isolate the absolute value term by subtracting 7 from each side
$\lvert 2x-5\rvert=-3$	✗ False

Careful! Observe the absolute value of $2x-5$ is a negative number. This is impossible with absolute value because *the result after evaluating absolute value must always be non-negative.* Thus, we say this equation has **no solution.**

1.2.5 Absolute Value Equations Homework

Solve each equation.

1. $|x| = 8$

2. $|b| = 1$

3. $|5 + 8a| = 53$

4. $|3k + 8| = 2$

5. $|9 + 7x| = 30$

6. $|8 + 6m| = 50$

7. $|6 - 2x| = 24$

8. $-7| - 3 - 3r| = -21$

9. $7| - 7x - 3| = 21$

10. $\dfrac{|-4b - 10|}{8} = 3$

11. $8|x + 7| - 3 = 5$

12. $5|3 + 7m| + 1 = 51$

13. $3 + 5|8 - 2x| = 63$

14. $|6b - 2| + 10 = 44$

15. $-7 + 8| - 7x - 3| = 73$

16. $|5x + 3| = |2x - 1|$

17. $|3x - 4| = |2x + 3|$

18. $\left|\dfrac{4x - 2}{5}\right| = \left|\dfrac{6x + 3}{2}\right|$

19. $|n| = 7$

20. $|x| = 2$

21. $|9n + 8| = 46$

22. $|3 - x| = 6$

23. $|5n + 7| = 23$

24. $|9p + 6| = 3$

25. $|3n - 2| = 7$

26. $|2 + 2b| + 1 = 3$

27. $\dfrac{|-4 - 3n|}{4} = 2$

28. $8|5p + 8| - 5 = 11$

29. $3 - |6n + 7| = -40$

30. $4|r + 7| + 3 = 59$

31. $5 + 8| - 10n - 2| = 101$

32. $7|10v - 2| - 9 = 5$

33. $8|3 - 3n| - 5 = 91$

34. $|2 + 3x| = |4 - 2x|$

35. $\left|\dfrac{2x - 5}{3}\right| = \left|\dfrac{3x + 4}{2}\right|$

36. $\dfrac{| - n + 6|}{6} = 0$

1.3 Literal equations

A literal equation is synonymous with a formula and similar to solving general linear equations because we apply the same method. We say, methods never change, just the problems. The only difference is we have several variables in the equation and we will attempt to solve for one specific variable of the formula. For example, we may have a formula such as $A = \pi r^2 + \pi rs$, the formula for surface area of a right circular cone, and we may be interested in solving for the variable s. This means we want to isolate the variable s so the equation has s isolated on one side, and everything else on the other. This looks like

$$s = \frac{A - \pi r^2}{\pi r}$$

This second equation gives the same information as the first, meaning they are algebraically equivalent. However, the original formula gives area, while the other gives s, the slant height of the cone. In this section, we discuss the process in which we start from the first equation and result in the second equation.

Example 1.39

Let's take a look at these two examples below, side by side. The left equation is a familiar one-step equation and the right equation is also a one-step equation, this time a literal equation (or formula).

$$\begin{array}{rrl}
3x = 12 & wx = z & \text{Both have coefficients} \\
 & & \text{Mutiply by the reciprocal of 3 and } w\text{, respectively} \\
\frac{1}{3} \cdot 3x = \frac{1}{3} \cdot 12 & \frac{1}{w} \cdot wx = z \cdot \frac{1}{w} & \text{Simplify} \\
x = 4 & x = \frac{z}{w} & \text{Solution}
\end{array}$$

We used the same process for solving $3x = 12$ for x as we did for solving $wx = z$ for x. Because we are solving for x, we treat all the other variables the same way we would treat numbers or coefficients. Thus, we applied the multiplication property and multiplied by the reciprocal of 3 and w to isolate x.

1.3.1 Solving for a variable with one and two-step equations

Example 1.40

Solve the equation $m + n = p$ for n.

Solution.

$$\begin{array}{rl}
m + n = p & \text{Add the opposite of } m \\
m + n + (-m) = p + (-m) & \text{Simplify} \\
n = p - m & \text{Solution}
\end{array}$$

Since p and m are not like terms, they cannot be combined. Hence, $n = p - m$.

Example 1.41

Solve the equation $a(x - y) = b$ for x.

Solution.

$$
\begin{aligned}
a(x-y) &= b && \text{Distribute}\\
ax - ay &= b && \text{Add the opposite of } ay\\
ax + ay + (-ay) &= b + (-ay) && \text{Simplify}\\
ax &= b - ay && \text{Isolate } x \text{ by multiplying by the reciprocal of } a\\
\frac{1}{a}\cdot ax &= (b-ay)\cdot\frac{1}{a} && \text{Simplify}\\
x &= \frac{b-ay}{a} && \text{Solution}
\end{aligned}
$$

Equivalently, x can be written as $\frac{b}{a} - y$ by simplifying the fraction. However, it is common practice to leave it as one fraction.

Example 1.42

Solve the equation $y = mx + b$ for m.

Solution.

$$
\begin{aligned}
y &= mx + b && \text{Isolate the variable term by adding the opposite of } b\\
y + (-b) &= mx + b + (-b) && \text{Simplify}\\
y - b &= mx && \text{Isolate } m \text{ by multiplying by the reciprocal of } x\\
\frac{1}{x}\cdot(y-b) &= mx\cdot\frac{1}{x} && \text{Simplify}\\
\frac{y-b}{x} &= m && \text{Rewrite with } m \text{ on the left side}\\
m &= \frac{y-b}{x} && \text{Solution}
\end{aligned}
$$

1.3.2 Solving for a variable in multiple steps

Example 1.43

Solve the equation $A = \pi r^2 + \pi rs$ for s. *This should remind you of the equation in the beginning of the section.*

Solution.

$$
\begin{aligned}
A &= \pi r^2 + \pi rs && \text{Isolate the variable term by adding the opposite of } \pi r^2\\
A + (-\pi r^2) &= \pi r^2 + \pi rs + (-\pi r^2) && \text{Simplify}\\
A - \pi r^2 &= \pi rs && \text{Isolate } s \text{ by multiplying by the reciprocal of } \pi r
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\pi r} \cdot \left(A - \pi r^2\right) &= \pi r s \cdot \frac{1}{\pi r} && \text{Simplify} \\
\frac{A - \pi r^2}{\pi r} &= s && \text{Rewrite with } s \text{ on the left side} \\
s &= \frac{A - \pi r^2}{\pi r} && \text{Solution}
\end{aligned}
$$

1.3.3 Solving for a variable with fractions

Formulas often include fractions and we can solve with the same method as used previously. First, identify the LCD, and then multiply each term by the LCD. After we clear denominators, we obtain a general equation and solve as usual.

Example 1.44

Solve the equation $h = \dfrac{2m}{n}$ for m.

Solution.

$$
\begin{aligned}
h &= \frac{2m}{n} && \text{Multiply by the LCD} = n \\
n \cdot h &= \frac{2m}{n} \cdot n && \text{Simplify} \\
nh &= 2m && \text{Multiply by the reciprocal of 2} \\
\frac{1}{2} \cdot nh &= 2m \cdot \frac{1}{2} && \text{Simplify} \\
\frac{nh}{2} &= m && \text{Rewrite with } m \text{ on the left side} \\
m &= \frac{nh}{2} && \text{Solution}
\end{aligned}
$$

Example 1.45

Solve the equation $\dfrac{a}{b} + \dfrac{c}{b} = e$ for a.

Solution.

$$
\begin{aligned}
\frac{a}{b} + \frac{c}{b} &= e && \text{Multiply each term by the LCD} = b \\
b \cdot \frac{a}{b} + b \cdot \frac{c}{b} &= e \cdot b && \text{Simplify} \\
a + c &= eb && \text{Add the opposite of } c \\
a + c + (-c) &= eb + (-c) && \text{Simplify} \\
a &= eb - c && \text{Solution}
\end{aligned}
$$

Example 1.46

Solve the equation $a = \dfrac{A}{2 - b}$ for b.

Solution.

$$\begin{aligned}
a &= \frac{A}{2-b} && \text{Multiply each term by the LCD} = (2-b) \\
(2-b)\cdot a &= \frac{A}{2-b}\cdot(2-b) && \text{Simplify} \\
a(2-b) &= A && \text{Distribute} \\
2a-2b &= A && \text{Isolate the variable term by adding the opposite of } 2a \\
2a-2b+(-2a) &= A+(-2a) && \text{Simplify} \\
-2b &= A-2a && \text{Multiply by the reciprocal of } -2 \\
-\frac{1}{2}\cdot -2b &= (A-2a)\cdot -\frac{1}{2} && \text{Simplify} \\
b &= -\frac{(A-2a)}{2} && \text{Distribute the negative} \\
b &= \frac{-A+2a}{2} && \text{Solution}
\end{aligned}$$

Note, we could also write the solution as $b = \dfrac{2a-A}{2}$, where the positive term is written first in the numerator. It's not necessary, but for aesthetic reasons, we can write b this way.

World Note

The father of algebra, Persian mathematician, *Muhammad ibn Musa Khwarizmi*, introduced the fundamental idea of balancing by subtracting the same term from the other side of the equation. He called this process *al-jabr*, which later became the world Algebra.

1.3.4 Literal Equations Homework

Solve each of the following equations for the indicated variable.

1. $ab = c$ for b

2. $\frac{f}{g}x = b$ for x

3. $3x = \frac{a}{b}$ for x

4. $E = mc^2$ for m

5. $V = \frac{4}{3}\pi r^3$ for π

6. $a + c = b$ for c

7. $c = \frac{4y}{m+n}$ for y

8. $V = \frac{\pi Dn}{12}$ for D

9. $P = n(p - c)$ for n

10. $T = \frac{D-d}{L}$ for D

11. $L = L_0(1 + at)$ for L_0

12. $2m + p = 4m + q$ for m

13. $\frac{k-m}{r} = q$ for k

14. $h = vt - 16t^2$ for v

15. $Q_1 = P(Q_2 - Q_1)$ for Q_2

16. $R = \frac{kA(T_1 + T_2)}{d}$ for T_1

17. $ax + b = c$ for a

18. $lwh = V$ for w

19. $\frac{1}{a} + b = \frac{c}{a}$ for a

20. $at - bw = s$ for t

21. $ax + bx = c$ for a

22. $x + 5y = 3$ for y

23. $3x + 2y = 7$ for y

24. $5a - 7b = 4$ for b

25. $4x - 5y = 8$ for y

26. $g = \frac{h}{i}$ for h

27. $p = \frac{3y}{q}$ for y

28. $\frac{ym}{b} = \frac{c}{d}$ for y

29. $DS = ds$ for D

30. $E = \frac{mv^2}{2}$ for m

31. $x - f = g$ for x

32. $\frac{rs}{a-3} = k$ for r

33. $F = k(R - L)$ for k

34. $S = L + 2B$ for L

35. $I = \frac{E_a - E_q}{R}$ for E_a

36. $ax + b = c$ for x

37. $q = 6(L - p)$ for L

38. $R = aT + b$ for T

39. $S = \pi rh + \pi r^2$ for h

40. $L = \pi(r_1 + r_2) + 2d$ for r_1

41. $P = \frac{V_1(V_2 - V_1)}{g}$ for V_2

42. $rt = d$ for r

43. $V = \frac{\pi r^2 h}{3}$ for h

44. $\frac{1}{a} + b = \frac{c}{a}$ for b

45. $at - bw = s$ for w

46. $x + 5y = 3$ for x

47. $3x + 2y = 7$ for x

48. $5a - 7b = 4$ for a

49. $4x - 5y = 8$ for x

50. $C = \frac{5}{9}(F - 32)$ for F

1.4 Word problems

Now, let's apply the techniques from this chapter to some common word problems. Word problems can be tricky. The goal is becoming proficient in translating an English sentence into a mathematical sentence. In this section, we focus on word problems modeled by a linear equation and solve. We discuss geometry problems including perimeter and triangles, number, and distance problems.

1.4.1 Number problems

Example 1.47

If 28 less than five times a number is 232, what is the number?

Solution.

First, let n be the number. Now, translate the key words in the sentence:

$$\ldots\underbrace{\text{28 less than }\underbrace{\text{five times a number}}_{5n}}_{5n-28}\ \underbrace{\text{is}}_{=}\ \underbrace{232}_{232}\ldots$$

Notice, after translating, we obtain the equation

$$5n - 28 = 232$$

Let's solve:

$$\begin{aligned} 5n - 28 &= 232 && \text{Isolate the variable term } 5n \\ 5n - 28 + 28 &= 232 + 28 && \text{Simplify} \\ 5n &= 260 && \text{Multiply by the reciprocal of 5} \\ \frac{1}{5} \cdot 5n &= 260 \cdot \frac{1}{5} && \text{Simplify} \\ n &= 52 && \text{Solution} \end{aligned}$$

Thus, the number is 52.

Example 1.48

Fifteen more than three times a number is the same as ten less than six times the number. What is the number?

Solution.

Notice, this sentence is a bit more challenging than example 1.47, but we still follow the method. Let n be the number.

$$\underbrace{\text{Fifteen more than }\underbrace{\text{three times a number}}_{3n}}_{3n+15}\ \underbrace{\text{is the same as}}_{=}\ \underbrace{\text{ten less than }\underbrace{\text{six times the number}}_{6n}}_{6n-10}$$

Notice, after translating, we obtain the equation

$$3n + 15 = 6n - 10$$

Let's solve:

$$\begin{aligned}
3n + 15 &= 6n - 10 && \text{Combine like terms} \\
3n + 15 + (-6n) &= 6n - 10 + (-6n) && \text{Simplify} \\
-3n + 15 &= -10 && \text{Isolate the variable term} \\
-3n + 15 + (-15) &= -10 + (-15) && \text{Simplify} \\
-3n &= -25 && \text{Multiply by the reciprocal of } -3 \\
-\frac{1}{3} \cdot -3n &= -25 \cdot -\frac{1}{3} && \text{Simplify} \\
n &= \frac{25}{3} && \text{Solution}
\end{aligned}$$

Thus, the number is $\frac{25}{3}$.

1.4.2 Consecutive integers

Another type of number problem involves *consecutive integers*.

Definition

Consecutive integers are integers that come one after the other (such as $3, 4, 5$, or $-3, -2, -1$).

- If we are trying to find several *consecutive integers*, it important to identify the first integer and then assign names to the following integers. E.g., if x is the first integer, then $x + 1$ will be the next, and $x + 2$ will be the following, and so on.
- If we are trying to find several *even or odd consecutive integers*, it important to identify the first integer and then assign names to the following even or odd integers. E.g., if x is the first integer, then $x + 2$ will be the next odd or even integer, and $x + 4$ will be the following, and so on.

Example 1.49

The sum of three consecutive positive integers is 93. What are the positive integers?

Solution.

Since we want to obtain three consecutive positive integers, then we can assign each integer as the following:

$$\begin{aligned}
x &\quad \text{is the first integer} \\
x + 1 &\quad \text{is the second integer} \\
x + 2 &\quad \text{is the third integer}
\end{aligned}$$

The sum of these three integers is given to be 93. Translating this into an equation, we get

$$x + (x + 1) + (x + 2) = 93$$

Let's solve this equation for x. Then we can obtain the other two integers.

$$\begin{aligned}
x+(x+1)+(x+2) &= 93 && \text{Rewrite without the parenthesis}\\
x+x+1+x+2 &= 93 && \text{Combine like terms}\\
3x+3 &= 93 && \text{Isolate the variable term}\\
3x+3+(-3) &= 93+(-3) && \text{Simplify}\\
3x &= 90 && \text{Multiply by the reciprocal of 3}\\
\frac{1}{3}\cdot 3x &= 90\cdot\frac{1}{3} && \text{Simplify}\\
x &= 30 && \text{First integer}
\end{aligned}$$

Since the first integer is 30, the next two integers would be

$$\begin{aligned}
30+1=31 &\quad \text{is the second even integer}\\
30+2=32 &\quad \text{is the third even integer}
\end{aligned}$$

Thus, the integers are 30, 31, and 32.

Example 1.50

The sum of three consecutive even positive integers is 246. What are the numbers?

Solution.

Since we want to obtain *three consecutive even positive integers*, then we can assign each integer as the following:

$$\begin{aligned}
x &\quad \text{is the first odd integer}\\
x+2 &\quad \text{is the second odd integer}\\
x+4 &\quad \text{is the third odd integer}
\end{aligned}$$

The sum of these three even integers is given to be 246. Translating this into an equation, we get

$$x+(x+2)+(x+4)=246$$

Let's solve this equation for x. Then we can obtain the other two integers.

$$\begin{aligned}
x+(x+2)+(x+4) &= 246 && \text{Rewrite without the parenthesis}\\
x+x+2+x+4 &= 246 && \text{Combine like terms}\\
3x+6 &= 246 && \text{Isolate the variable term}\\
3x+6+(-6) &= 246+(-6) && \text{Simplify}\\
3x &= 240 && \text{Multiply by the reciprocal of 3}\\
\frac{1}{3}\cdot 3x &= 240\cdot\frac{1}{3} && \text{Simplify}\\
x &= 80 && \text{First integer}
\end{aligned}$$

Since the first integer is 80, the next two even integers would be

$$\begin{aligned}
80+2=82 &\quad \text{is the second even integer}\\
80+4=84 &\quad \text{is the third even integer}
\end{aligned}$$

Thus, the integers are 80, 82, and 84.

Example 1.51

Find three consecutive odd positive integers so that the sum of twice the first integer, the second integer, and three times the third integer is 152.

Solution.

Since we want to obtain *three consecutive odd positive integers*, then we can assign each integer as the following:

$$\begin{aligned} x &\quad \text{is the first odd integer} \\ x+2 &\quad \text{is the second odd integer} \\ x+4 &\quad \text{is the third odd integer} \end{aligned}$$

The sum of twice the first integer, the second integer, and three times the third integer is given to be 152. Translating this into an equation, we get

$$2 \cdot x + (x+2) + 3 \cdot (x+4) = 152$$

Let's solve this equation for x. Then we can obtain the other two integers.

$$\begin{aligned} 2 \cdot x + (x+2) + 3 \cdot (x+4) &= 152 && \text{Rewrite without the parenthesis} \\ 2x + x + 2 + 3x + 12 &= 152 && \text{Combine like terms} \\ 6x + 14 &= 152 && \text{Isolate the variable term} \\ 6x + 14 + (-14) &= 152 + (-14) && \text{Simplify} \\ 6x &= 138 && \text{Multiply by the reciprocal of 6} \\ \frac{1}{6} \cdot 6x &= 138 \cdot \frac{1}{6} && \text{Simplify} \\ x &= 23 && \text{First integer} \end{aligned}$$

Since the first integer is 23, the next two odd integers would be

$$\begin{aligned} 23 + 2 = 25 &\quad \text{is the second odd integer} \\ 23 + 4 = 27 &\quad \text{is the third odd integer} \end{aligned}$$

Thus, the integers are 23, 25, and 27.

1.4.3 Perimeter problems

Another problem from geometry involves perimeter or the distance around an object.

Perimeter of a rectangle

The formula for the perimeter of a rectangle is given by

$$P = 2w + 2\ell,$$

where w is the width and ℓ is the length of the rectangle.

Example 1.52

The perimeter of a rectangle is 44 cm. The length is 5 less than double the width. Find the dimensions.

Solution.

Let w be the width of the rectangle. Then the length is $2w - 5$. Since the perimeter is 44 cm, the we can use the perimeter formula to obtain the dimensions.

$$\begin{aligned} P &= 2w + 2\ell && \text{Substitute in the width, length, and perimeter} \\ 44 &= 2(w) + 2(2w-5) && \text{Rewrite with no parenthesis} \\ 44 &= 2w + 4w - 10 && \text{Combine like terms} \\ 44 &= 6w - 10 && \text{Isolate the variable term} \\ 54 &= 6w && \text{Multiply by the reciprocal of 6} \\ 9 &= w && \text{Length of the rectangle} \end{aligned}$$

Since the width is 9 cm, then the length is $(2(9) - 5) = 13$ cm.

1.4.4 Triangles

Sum of Angles in a Triangle

Given a triangle, the sum of the three angles is 180°. I.e., if the angles in a triangle are $a°$, $b°$, and $c°$, then

$$a° + b° + c° = 180°$$

World Note

German mathematician *Bernhart Thibaut* in 1809 tried to prove that the angles of a triangle add to 180 without using Euclid's parallel postulate (a point of much debate in math history). He created a proof, but it was later shown to have an error in the proof.

Example 1.53

The second angle of a triangle is double the first. The third angle is 40 less than the first. Find the three angles.

Solution.

Let x be the measure of the first angle. Then

$$\begin{aligned} 2x &\quad \text{is the measure of the second angle} \\ x-40 &\quad \text{is the measure of the third angle} \end{aligned}$$

Since the sum of these three angles is $180°$, then we can write the equation

$$x + 2x + (x - 40) = 180$$

Let's solve for the first angle x:

$$\begin{aligned} x + 2x + (x - 40) &= 180 &&\text{Rewrite without parenthesis} \\ x + 2x + x - 40 &= 180 &&\text{Combine like terms} \\ 4x - 40 &= 180 &&\text{Isolate the variable term} \\ 4x &= 220 &&\text{Multiply by the reciprocal of 4} \\ x &= 55 &&\text{Measure of the first angle} \end{aligned}$$

Since the measure of the first angle is $55°$, then the measures of the second and third angle are

$$\begin{aligned} 2(55) = 110° &\text{ is the measure of the second angle} \\ 55 - 40 = 15° &\text{ is the measure of the third angle} \end{aligned}$$

1.4.5 Uniform motion problems

Another common application of linear equations is uniform motion problems. When solving uniform motion problems, we use the relationship $rt = d$ or

$$\text{rate (speed)} \cdot \text{time} = \text{distance}$$

For example, if a person were to travel 30 miles per hour (mph) for 4 hours, to find the total distance we would multiply rate and the time: $(30)(4) = 120$. Hence, this person traveled a distance of 120 miles. The problems we solve in this section are just a few more steps than described. To keep the information in the problem organized, we use tables.

Opposite directions

Example 1.54

Two joggers start from opposite ends of an 8 mile course running towards each other. One jogger is running at a rate of 4 miles per hour, and the other is running at a rate of 6 miles per hour. After how long will the joggers meet?

Solution.

First, we can make a table to organize the given information and then create an equation. Let t represent the length of time until the joggers meet.

	rate	**time**	**distance**
Jogger 1	4	t	$4t$
Jogger 2	6	t	$6t$

Now we can set up the equation. If the total distance is 8 miles, then

$$4t + 6t = 8,$$

i.e., the sum of Jogger 1's distance and Jogger 2's distance is 8 miles. Let's solve.

$$\begin{aligned} 4t + 6t &= 8 && \text{Combine like terms} \\ 10t &= 8 && \text{Multiply by the reciprocal of 10} \\ t &= \frac{4}{5} && \text{Hours until they meet} \end{aligned}$$

It will be $\frac{4}{5}$ hours (or 48 minutes) until they meet.

Example 1.55

Bob and Fred start from the same point and walk in opposite directions. Bob walks 2 miles per hour faster than Fred. After 3 hours they are 30 miles apart. How fast did each walk?

Solution.

First, we can make a table to organize the given information and then create an equation. Let r represent the rate of Fred.

	rate	**time**	**distance**
Bob	$r + 2$	3	$3(r + 2)$
Fred	r	3	$3r$

Now we can set up the equation. If the total distance is 30 miles, then

$$3(r + 2) + 3r = 30,$$

i.e., the sum of Bob's distance and Fred's distance is 30 miles. Let's solve.

$$\begin{aligned} 3(r + 2) + 3r &= 30 && \text{Distribute} \\ 3r + 6 + 3r &= 30 && \text{Combine like terms} \\ 6r + 6 &= 30 && \text{Isolate the variable term} \\ 6r &= 24 && \text{Multiply by the reciprocal of 6} \\ r &= 4 && \text{Rate of Fred} \end{aligned}$$

Since the rate of Fred is 4 mph, then Bob's rate is 6 mph $(4 + 2 = 6)$.

Example 1.56

Two campers left their campsite by canoe and paddled downstream at an average speed of 12 miles per hour. They turned around and paddled back upstream at an average rate of 4 miles per hour. The total trip took 1 hour. After how much time did the campers turn around downstream?

Solution.

First, we can make a table to organize the given information and then create an equation. Let t

represent the time it took to travel upstream.

	rate	time	distance
upstream	4	t	$4t$
downstream	12	$1-t$	$12(1-t)$

Now we can set up the equation. If the upstream and downstream routes' distances are the same, then

$$4t = 12(1-t)$$

Let's solve.

$$\begin{aligned} 4t &= 12(1-t) && \text{Distribute} \\ 4t &= 12 - 12t && \text{Combine like terms} \\ 16t &= 12 && \text{Multiply by the reciprocal of 16} \\ t &= \frac{12}{16} && \text{Reduce} \\ t &= \frac{3}{4} && \text{Time going upstream} \end{aligned}$$

Since the time going upstream is $\frac{3}{4}$ hours, then downstream's time is $\frac{1}{4}$ hours $\left(1 - \frac{3}{4} = \frac{1}{4}\right)$. Thus, the campers spent 15 minutes going downstream.

Catch-up

Example 1.57

Mike leaves his house traveling 2 miles per hour. Joy leaves 6 hours later to catch up with him traveling 8 miles per hour. How long will it take her to catch up with him?

Solution.

First, we can make a table to organize the given information and then create an equation. Let t represent the time Joy traveled.

	rate	time	distance
Mike	2	$t+6$	$2(t+6)$
Joy	8	t	$8t$

Now we can set up the equation. If Joy catches up to Mike, then Mike and Joy would have traveled the same distance. Hence, giving the equation

$$2(t+6) = 8t,$$

i.e., Mike's distance and Joy's distance are the same. Let's solve.

$$\begin{aligned} 2(t+6) &= 8t && \text{Distribute} \\ 2t + 12 &= 8t && \text{Combine like terms} \\ 12 &= 6t && \text{Multiply by the reciprocal of 6} \\ 2 &= t && \text{Time Joy traveled} \end{aligned}$$

Since the time Joy traveled was 2 hours, then Mike traveled 8 hours $(2 + 6 = 8)$. Thus, it took 2 hours for Joy to catch up with Mike.

World Note

The 10,000-meter race is the longest standard track event. Ten-thousand meters is approximately 6.2 miles. The current (at the time of printing) world record for this race is held by Ethiopian *Kenenisa Bekele* with a time of 26 minutes, 17.53 seconds. That is a rate of 12.7 miles per hour.

Total time

Example 1.58

On a 130-mile trip, a car traveled at an average speed of 55 mph and then reduced its speed to 40 mph for the remainder of the trip. The trip took 2.5 hours. For how long did the car travel 40 mph?

Solution.

First, we can make a table to organize the given information and then create an equation. Let t represent the time the car traveled at the faster speed.

	rate	time	distance
First part	55	t	$55t$
Second part	40	$2.5 - t$	$40(2.5 - t)$

Now we can set up the equation. Since the total distance of the trip was 130 miles, then

$$55t + 40(2.5 - t) = 130,$$

i.e., the sum of the first part's distance and the second part's distance is 130 miles. Let's solve.

$$\begin{aligned}
55t + 40(2.5 - t) &= 130 && \text{Distribute}\\
55t + 100 - 40t &= 130 && \text{Combine like terms}\\
15t + 100 &= 130 && \text{Isolate the variable term}\\
15t &= 30 && \text{Multiply by the reciprocal of 15}\\
t &= 2 && \text{First part's travel time}
\end{aligned}$$

Since the first part of the trip took 2 hours, then the car traveled 0.5 hours (or 30 minutes) at 40 mph.

1.4.6 Word Problems Homework

1. When five is added to three more than a certain number, the result is 19. What is the number?

2. If five is subtracted from three times a certain number, the result is 10. What is the number?

3. When 18 is subtracted from six times a certain number, the result is -42. What is the number?

4. A certain number added twice to itself equals 96. What is the number?

5. A number plus itself, plus twice itself, plus 4 times itself, is equal to -104. What is the number?

6. Sixty more than nine times a number is the same as two less than ten times the number. What is the number?

7. Eleven less than seven times a number is five more than six times the number. Find the number.

8. Fourteen less than eight times a number is three more than four times the number. What is the number?

9. The sum of three consecutive integers is 108. What are the integers?

10. The sum of three consecutive integers is -126. What are the integers?

11. Find three consecutive integers such that the sum of the first, twice the second, and three times the third is -76.

12. The sum of two consecutive even integers is 106. What are the integers?

13. The sum of three consecutive odd integers is 189. What are the integers?

14. The sum of three consecutive odd integers is 255. What are the integers?

15. Find three consecutive odd integers such that the sum of the first, two times the second, and three times the third is 70.

16. The second angle of a triangle is the same size as the first angle. The third angle is 12 degrees larger than the first angle. How large are the angles?

17. Two angles of a triangle are the same size. The third angle is 12 degrees smaller than the first angle. Find the measure the angles.

18. Two angles of a triangle are the same size. The third angle is 3 times as large as the first. How large are the angles?

19. The third angle of a triangle is the same size as the first. The second angle is 4 times the third. Find the measure of the angles.

20. The second angle of a triangle is 3 times as large as the first angle. The third angle is 30 degrees more than the first angle. Find the measure of the angles.

21. The second angle of a triangle is twice as large as the first. The measure of the third angle is 20 degrees greater than the first. How large are the angles?

22. The second angle of a triangle is three times as large as the first. The measure of the third angle is 40 degrees greater than that of the first angle. How large are the three angles?

23. The second angle of a triangle is five times as large as the first. The measure of the third angle is 12 degrees greater than that of the first angle. How large are the angles?

24. The second angle of a triangle is three times the first, and the third is 12 degrees less than twice the first. Find the measures of the angles.

25. The second angle of a triangle is four times the first and the third is 5 degrees more than twice the first. Find the measures of the angles.

26. The perimeter of a rectangle is 150 cm. The length is 15 cm greater than the width. Find the dimensions.

27. The perimeter of a rectangle is 304 cm. The length is 40 cm longer than the width. Find the length and width.

28. The perimeter of a rectangle is 152 meters. The width is 22 meters less than the length. Find the length and width.

29. The perimeter of a rectangle is 280 meters. The width is 26 meters less than the length. Find the length and width.

30. The perimeter of a college basketball court is 96 meters and the length is 14 meters more than the width. What are the dimensions?

31. A is 60 miles from B. An automobile at A starts for B at the rate of 20 miles per hour at the same time that an automobile at B starts for A at the rate of 25 miles an hour. How long will it be before the automobiles meet?

32. Two automobiles are 276 miles apart and start at the same time to travel toward each other. They travel at rates differing by 5 miles per hour. If they meet after 6 hours, find each rate.

33. Two trains travel toward each other from points which are 195 miles apart. They travel at rate of 25 and 40 miles an hour, respectively. If they start traveling at the same time, how long before the trains will meet?

34. Car A and Car B start traveling towards each other at the same time from points 150 miles apart. If Car A went at the rate of 20 miles an hour, at what rate must B travel if they meet in 5 hours?

35. A passenger and a freight train start toward each other at the same time from two points 300 miles apart. If the rate of the passenger train exceeds the rate of the freight train by 15 miles per hour, and they meet after 4 hours, what are the rates of the passenger and train?

36. Two automobiles started at the same time from a point, but traveled in opposite directions. Their rates were 25 and 35 miles per hour, respectively. After how many hours were they 180 miles apart?

37. A man having ten hours at his disposal made an excursion, riding out at the rate of 10 miles an hour and returning on foot at the rate of 3 miles an hour. Find the distance he rode.

38. A man walks at the rate of 4 miles per hour. How far can he walk into the country and ride back on a trolley that travels at the rate of 20 miles per hour if he must be back home 3 hours from the time he started?

39. A boy rides away from home in an automobile at the rate of 28 miles an hour and walks back at the rate of 4 miles an hour. The round trip requires 2 hours. How far does he ride in the automobile?

40. A motorboat leaves a harbor and travels at an average speed of 15 mph toward an island. The average speed on the return trip was 10 mph. How far was the island from the harbor if the total trip took 5 hours?

41. A family drove to a resort at an average speed of 30 mph and later returned over the same road at an average speed of 50 mph. Find the distance to the resort if the total driving time was 8 hours.

42. As part of his flight training, a student pilot was required to fly to an airport and then return. The average speed to the airport was 90 mph, and the average speed returning was 120 mph. Find the distance between the two airports if the total flying time was 7 hours.

43. Annie, who travels 4 miles an hour starts from a certain place 2 hours in advance of Brandie, who travels 5 miles an hour in the same direction. How many hours must Brandie travel to overtake Annie?

44. A man travels 5 miles an hour. After traveling for 6 hours another man starts at the same place following the first man at the rate of 8 miles an hour. When will the second man overtake the first man?

45. A motorboat leaves a harbor and travels at an average speed of 8 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 16 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cruiser be alongside the motorboat?

46. A long distance runner started on a course running at an average speed of 6 mph. One hour later, a second runner began the same course at an average speed of 8 mph. How long after the second runner started will the second runner overtake the first runner?

47. A car traveling at 48 mph overtakes a cyclist who, riding at 12 mph, has had a 3-hour head start. How far from the starting point does the car overtake the cyclist?

48. A jet plane traveling at 600 mph overtakes a propeller-driven plane which has had a 2-hour head start. The propeller-driven plane is traveling at 200 mph. How far from the starting point does the jet overtake the propeller-driven plane?

49. Two men are traveling in opposite directions at the rate of 20 and 30 miles per hour at the same time and from the same place. In how many hours will they be 300 miles apart?

50. Running at an average rate of 8 meters per second, a sprinter ran to the end of a track and then jogged back to the starting point at an average rate of 3 meters per second. The sprinter took 55 seconds to run to the end of the track and jog back. Find the length of the track.

51. A motorboat leaves a harbor and travels at an average speed of 18 mph to an island. The average speed on the return trip was 12 mph. How far was the island from the harbor if the total trip took 5 hours?

52. A motorboat leaves a harbor and travels at an average speed of 9 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 18 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cruiser be alongside the motorboat?

53. A jet plane traveling at 570 mph overtakes a propeller-driven plane that has had a 2-hour head start. The propeller-driven plane is traveling at 190 mph. How far from the starting point does the jet overtake the propeller-driven plane?

54. Two trains start at the same time from the same place and travel in opposite directions. If the rate of one is 6 miles per hour more than the rate of the other and they are 168 miles apart at the end of 4 hours, what is each rate?

55. As part of flight training, a student pilot was required to fly to an airport and then return. The average speed on the way to the airport was 100 mph, and the average speed returning was 150 mph. Find the distance between the two airports if the total flight time was 5 hours.

56. Two cyclists start from the same point and ride in opposite directions. One cyclist rides twice as fast as the other. In three hours they are 72 miles apart. Find the rate of each cyclist.

57. A car traveling at 56 mph overtakes a cyclist who, riding at 14 mph, has had a 3-hour head start. How far from the starting point does the car overtake the cyclist?

58. Two small planes start from the same point and fly in opposite directions. The first plane is flying 25 mph slower than the second plane. In two hours, the planes are 430 miles apart. Find the rate of each plane.

59. A bus traveling at a rate of 60 mph overtakes a car traveling at a rate of 45 mph. If the car had a 1-hour head start, how far from the starting point does the bus overtake the car?

60. Two small planes start from the same point and fly in opposite directions. The first plane is flying 25 mph slower than the second plane. In 2 hours, the planes are 470 mi apart. Find the rate of each plane.

61. A truck leaves a depot at 11 A.M. and travels at a speed of 45 mph. At noon, a van leaves the same place and travels the same route at a speed of 65 mph. At what time does the van overtake the truck?

62. A family drove to a resort at an average speed of 25 mph and later returned over the same road at an average speed of 40 mph. Find the distance to the resort if the total driving time was 13 hours.

63. Three campers left their campsite by canoe and paddled downstream at an average rate of 10 mph. They then turned around and paddled back upstream at an average rate of 5 mph to return to their campsite. How long did it take the campers to canoe downstream if the total trip took 1 hour?

64. A motorcycle breaks down and the rider has to walk the rest of the way to work. The motorcycle was being driven at 45 mph, and the rider walks at a speed of 6 mph. The distance from home to work is 25 miles, and the total time for the trip was 2 hours. How far did the motorcycle go before it broke down?

65. A student walks and jogs to college each day. The student averages 5 kilometers per hour walking and 9 kilometers per hour jogging. The distance from home to college is 8 kilometers, and the student makes the trip in one hour. How far does the student jog?

66. On a 130-mile trip, a car traveled at an average speed of 55 mph and then reduced its speed to 40 mph for the remainder of the trip. The trip took a total of 2.5 hours. For how long did the car travel at 40 mph?

67. On a 220-mile trip, a car traveled at an average speed of 50 mph and then reduced its average speed to 35 mph for the remainder of the trip. The trip took a total of 5 hours. How long did the car travel at each speed?

68. An executive drove from home at an average speed of 40 mph to an airport where a helicopter was waiting. The executive boarded the helicopter and flew to the corporate offices at and average speed of 60 mph. The entire distance was 150 miles. The entire trip took 3 hours. Find the distance from the airport to the corporate offices.

1.5 Linear Equations: Answers to the Homework Exercises

Linear Equations

1. $v = 7$

3. $a = 10$

5. $n = 18$

7. $n = -108$

9. $n = 17$

11. $n = 3$

13. $x = 15$

15. $r = 5$

17. $x = 14$

19. $p = -240$

21. $b = 11$

23. $k = 6$

25. $m = 6$

27. $b = 5$

29. $a = 4$

31. $x = 16$

33. $n = 8$

35. $n = 2$

37. $x = 1$

39. $x = -135$

41. $n = -4$

43. $n = 10$

45. $x = -10$

47. $x = 4$

49. $r = 7$

51. $n = 11$

53. $k = 1$

55. $p = -6$

57. $r = 1$

59. $v = -12$

61. $m = 7$

63. $b = -12$

65. $a = -16$

67. $m = -5$

69. $n = 12$

71. $v = -6$

73. $n = 4$

75. $x = 6$

77. $x = -13$

79. $x = 0$

81. $a = -3$

83. $x = 1$

85. $x = 0$

87. $m = 8$

89. $b = 2$

91. $m = 3$

93. $v = 8$

95. $a = -1$

97. $m = -3$

99. $n = -6$

101. $n = 0$

103. $x = 12$

105. $p = -9$

107. $x = 0$

109. $k = -4$

111. all real numbers

113. $x = 0$

115. $x = 3$

117. $b = -1$

119. $n = 5$

121. $x = 0$

123. $x = -2$

125. $x = 6$

127. $x = 0$

129. $x = 1$

131. $p = \frac{3}{4}$

133. $m = -\frac{19}{6}$

135. $b = -2$

137. $a = -\frac{3}{2}$

139. $n = 0$

141. $b = \frac{1}{2}$

143. $n = 16$

145. $x = \frac{4}{3}$

147. $n = \frac{1}{6}$

149. $n = -\frac{1}{3}$

151. $x = \frac{4}{3}$

153. $x = 1$

155. $x = 1$

157. $r = -\frac{9}{4}$

159. $x = -\frac{3}{2}$

Absolute Value Equations

1. $8, -8$

3. $6, -\frac{29}{4}$

5. $3, -\frac{39}{7}$

7. $-9, 15$

9. $-\frac{6}{7}, 0$

11. $-6, -8$

13. $-2, 10$

15. $-\frac{13}{7}, 1$

17. $7, \frac{1}{5}$

19. $7, -7$

21. $\frac{38}{9}, -6$

23. $\frac{16}{5}, -6$

25. $3, -\frac{5}{3}$

27. $-4, \frac{4}{3}$

29. $6, -\frac{25}{3}$

31. $-\frac{7}{5}, 1$

33. $-3, 5$

35. $-\frac{22}{5}, -\frac{2}{13}$

Literal Equations

1. $b = \frac{c}{a}$

3. $x = \frac{a}{3b}$

5. $\pi = \frac{3V}{4r^3}$

7. $y = \frac{cm + cn}{4}$

9. $n = \frac{P}{p - c}$

11. $L_0 = \frac{L}{1 + at}$

13. $k = qr + m$

15. $Q_2 = \frac{Q_1 + PQ_1}{P}$

17. $a = \frac{c - b}{x}$

19. $a = \frac{c - 1}{b}$

21. $a = \frac{c - bx}{x}$

23. $y = \dfrac{7-3x}{2}$

25. $y = \dfrac{4x-8}{5}$

27. $y = \dfrac{pq}{3}$

29. $D = \dfrac{ds}{S}$

31. $x = f + g$

33. $k = \dfrac{F}{R-L}$

35. $E_a = IR + E_q$

37. $L = \dfrac{q+6p}{6}$

39. $h = \dfrac{s-\pi r^2}{\pi r}$

41. $V_2 = \dfrac{Pg + {V_1}^2}{V_1}$

43. $h = \dfrac{3v}{\pi r^2}$

45. $w = \dfrac{at-s}{b}$

47. $x = \dfrac{7-2y}{3}$

49. $x = \dfrac{8+5y}{4}$

Word Problems

1. 11

3. -4

5. -13

7. 16

9. 35, 36, 37

11. -14, -13, -12

13. 61, 63, 65

15. 9, 11, 13

17. 56, 56, 68

19. 36, 36, 108

21. 30, 90, 60

23. 28, 84, 68

25. 32, 96, 52

26. 45, 30

28. 27, 49

30. 17, 31

31. $1\frac{1}{3}$

33. 3

35. 30, 45

37. $\frac{300}{13}$

39. 7

41. 150

43. 8

45. 2

47. 48

49. 6

51. 36

53. 570

55. 300

57. 56

59. 180

61. 2:15 PM

63. $\frac{1}{3}$

65. $\frac{27}{4}$

67. 3, 2

Chapter 2

Graphing Linear Equations

Chapter Objectives

By the end of this chapter, the student should be able to

- Find the slope of a line from two points or a graph
- Find the equation of a line from its graph, the standard form, two given points
- Obtain equations of parallel and perpendicular lines

2.1 Graphing and slope

Often, to get an idea of the behavior of an equation, we will make a picture that represents the solutions to the equations called a *graph*. Let's have an introduction to graphing so that later we can construct graphs.

Definition

A ***graph*** represents a visualization of the solutions to an equation.

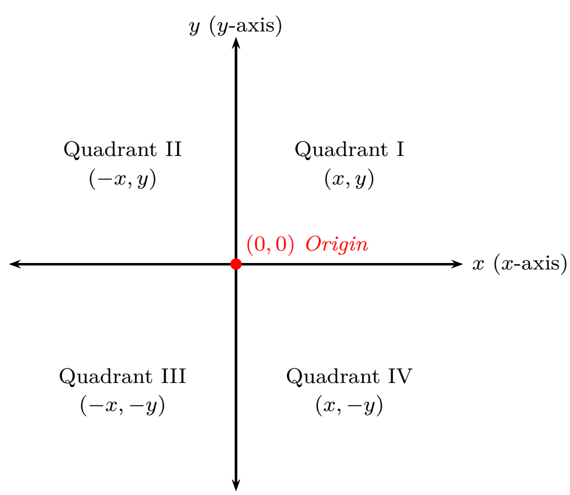


Figure 2.1: Coordinate plane

Coordinate plane

- The ***coordinate plane*** is divided into four quadrants by a horizontal real number line, called the x-axis, and a vertical real number line, called the y-axis.
- The two axes meet at the ***origin***, where $x = 0$ and $y = 0$.
- The ***x-coordinate*** is the horizontal distance from the origin and the ***y-coordinate*** is the vertical distance from the origin.
- An ***ordered-pair*** is where a point in the coordinate plane is located and denoted by (x, y).
- The quadrants are labeled in counter-clockwise starting at the top right.

World Note

Locations on the globe are similar to ordered-pairs. Each number is a distance from a central point, the origin, located where the prime meridian and the equator meet. This "origin" is just off the western coast of Africa.

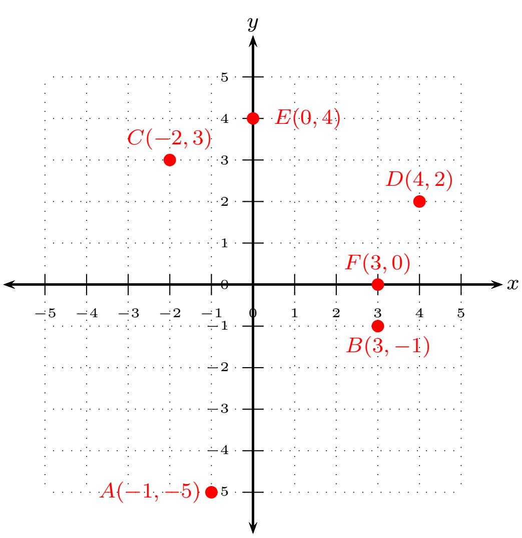
2.1.1 Points and lines

Example 2.1

Plot each ordered-pair and identify the quadrant in which lies the ordered pair:

$$A(-1,-5), B(3,-1), C(-2,3), D(4,2), E(0,4), F(3,0)$$

Solution.



- For point $A(-1,-5)$, notice the x-coordinate is -1. Since the x-coordinate is the horizontal distance from the origin, then we move 1 unit to the left. Looking at the y-coordinate, -5, we see this will be the vertical distance. Hence, we will move 5 units downward from the origin. Starting at the origin, move one unit left, then 5 units down. Point A is in quadrant III.
- For point $B(3,-1)$, notice the x-coordinate is 3. Since the x-coordinate is the horizontal distance from the origin, then we move 3 units to the right. Looking at the y-coordinate, -1, we see this will be the vertical distance. Hence, we will move one unit downward from the origin. Starting at the origin, move 3 units right, then 1 unit down. Point B is in quadrant IV.
- For point $C(-2,3)$, notice the x-coordinate is -2. Since the x-coordinate is the horizontal distance from the origin, then we move 2 units to the left. Looking at the y-coordinate, 3, we see this will be the vertical distance. Hence, we will move 3 units upward from the origin. Starting at the origin, move 2 units left, then 3 units up. Point C is in quadrant III.
- For point $D(4,2)$, notice the x-coordinate is 4. Since the x-coordinate is the horizontal

distance from the origin, then we move 4 units to the right. Looking at the y-coordinate, 2, we see this will be the vertical distance. Hence, we will move 2 units upward from the origin. Starting at the origin, move 4 units right, then 2 units up. Point D is in quadrant I.

- For point $E(0,4)$, notice the x-coordinate is 0. Since the x-coordinate is the horizontal distance from the origin, then we move no units horizontally from the origin. Looking at the y-coordinate, 4, we see this will be the vertical distance. Hence, we will move 4 units upward from the origin. Starting at the origin, move 4 units up. Point E is not in any quadrant as it lies on the y-axis.
- For point $F(3,0)$, notice the x-coordinate is 3. Since the x-coordinate is the horizontal distance from the origin, then we move 3 units to the right. Looking at the y-coordinate, 0, we see this will be the vertical distance. Hence, we will move no units vertically from the origin. Starting at the origin, move 3 units right. Point F is not in any quadrant as it lies on the x-axis.

Note

Notice, in points A, B, C, the negative coordinates didn't imply negative distance from the origin. The *negative on these coordinates implies the direction* in which we move: horizontal- we move left or right, vertical- we move up or down. If the x-coordinate is negative, then we move to the left. If the y-coordinate is negative, then we move downward.

Definition

From example 2.1, with points E and F, we could see that these points did not lie in a quadrant, but on an axis. These are special points on graphs and are called ***intercepts***.

- The ***x-intercept*** of a graph is the point(s) where the graph crosses the x-axis, i.e., $y = 0$.
- The ***y-intercept*** of a graph is the point(s) where the graph crosses the y-axis, i.e., $x = 0$.

Example 2.2

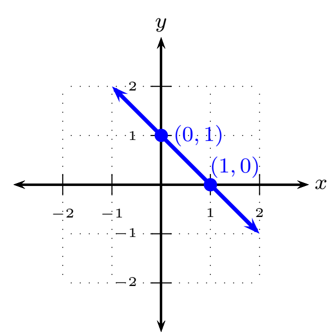
Graph $y = 1 - x$ by plotting the intercepts.

Solution.

To find the x and y-intercepts, we can follow the definition above and find where $y = 0$ and $x = 0$, respectively. Let's make a table.

x	$y = 1 - x$	(x, y)
$0 = 1 - x \implies x = 1$	0	$(1, 0)$
0	$y = 1 - 0 = 1$	$(0, 1)$

We can see when $y = 0$, $x = 1$ since $0 = 1 - x$ only when $x = 1$. Let's plot the two intercepts from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Draw the line to fill the grid and put arrows at the ends. It is recommended to purchase a small 6-inch ruler to make nice straight lines.



The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$ and be interested in the type of solutions that are possible for this equation. We can visualize the solution by making a graph of possible x and y combinations that makes this equation a true statement. We have to start by finding possible x and y combinations. We do this by using a table of values.

Example 2.3

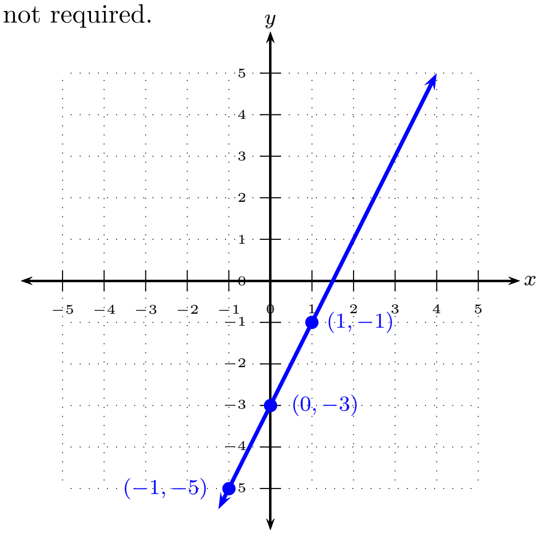
Graph $y = 2x - 3$ by plotting points, i.e., by making a T-table.

Solution.

Usually, we pick three x-coordinates, and find corresponding y-values. Each x-value being positive, negative, and zero. This is common practice, but not required.

x	$y = 2x - 3$	(x, y)
-1	$y = 2(-1) - 3 = -2 - 3 = -5$	$(-1, -5)$
0	$y = 2(0) - 3 = 0 - 3 = -3$	$(0, -3)$
1	$y = 2(1) - 3 = 2 - 3 = -1$	$(1, -1)$

Plot the three ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Draw the line to fill the grid and put arrows at the ends. It is recommended to purchase a small 6-inch ruler to make nice straight lines.



Example 2.4

Graph $2x - 3y = 6$ by plotting points, i.e., by making a T-table.

Solution.

Let's begin by choosing x-values for the table. Notice this equation isn't as simple as the prior example, so we will have to do a bit of algebra to solve for the y-value. Then fill in the table.

x	y	(x, y)
-3		
0		
3		

Let's evaluate $2x - 3y = 6$ for each of the chosen x-values:

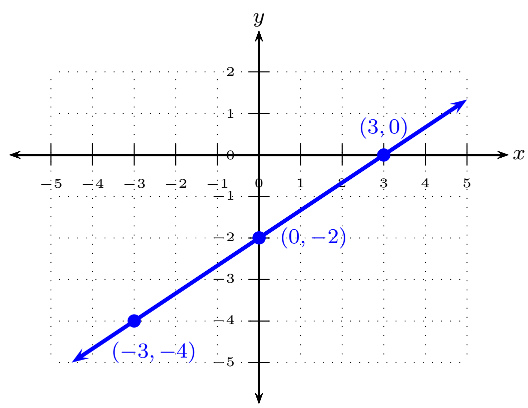
$$x = -3: \qquad \begin{aligned} 2(-3) - 3y &= 6 \\ -6 - 3y + 6 &= 6 + 6 \\ -\frac{1}{3} \cdot -3y &= 12 \cdot -\frac{1}{3} \\ y &= -4 \end{aligned}$$

$$x = 0: \qquad \begin{aligned} 2(0) - 3y &= 6 \\ 0 - 3y &= 6 \\ -\frac{1}{3} \cdot -3y &= 6 \cdot -\frac{1}{3} \\ y &= -2 \end{aligned}$$

$$x = 3: \qquad \begin{aligned} 2(3) - 3y &= 6 \\ 6 - 3y + (-6) &= 6 + (-6) \\ -\frac{1}{3} \cdot -3y &= 0 \cdot -\frac{1}{3} \\ y &= 0 \end{aligned}$$

Now, we fill in the table with y-values and ordered-pairs, and then graph $2x - 3y = 6$.

x	y	(x, y)
-3	-4	$(-3, -4)$
0	-2	$(0, -2)$
3	0	$(3, 0)$



2.1.2 Obtaining the slope of a line from its graph

As we graph lines, we want to identify different properties of lines. One of the most important properties of a line is its slope.

Definition

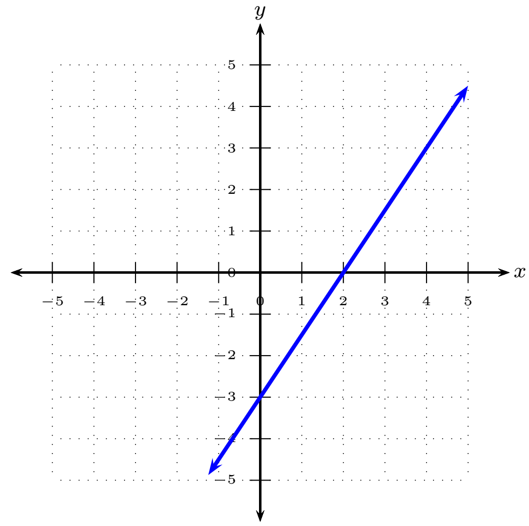
The ***slope*** of a line is the measure of the line's steepness.

- We denote slope with m. One theory from mathematicians that began working with slope was that it was called the *modular slope*.
- As $|m|$ increases, the line becomes steeper. As $|m|$ decreases, the line becomes flatter.
- A line that rises left to right has a positive slope and and a line that falls left to right has negative slope.
- m is the change of y divided by the change in x, i.e.,

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\textit{rise}}{\textit{run}}$$

Example 2.5

Find the slope from the graph given of the line.

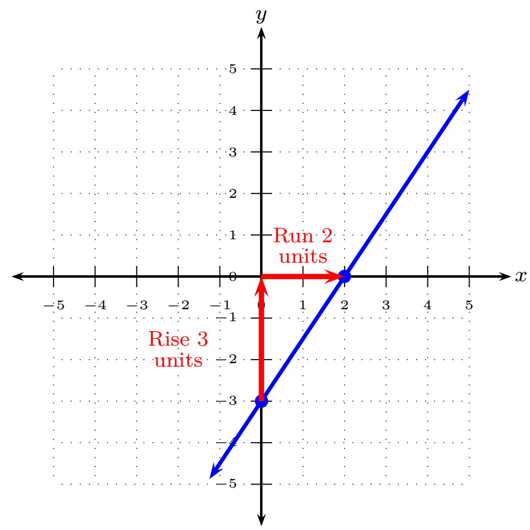


Solution.

We start at a well-defined point, preferably a point on the y-axis, i.e., the y-intercept. Then count the number of units we rise (up/down) and run (left/right) to reach the next well-defined point. We will start at $(0, -3)$ and reach the next point, $(2, 0)$. Notice we rise upward 3 units and run to the right 2 units. Using the ratio, $m = \dfrac{rise}{run}$, we get the rise to be 3 and the run to be 2:

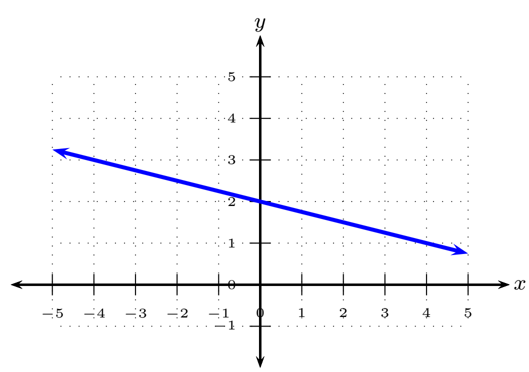
$$m = \frac{rise}{run} = \frac{3}{2}$$

Thus, the slope is $\dfrac{3}{2}$.



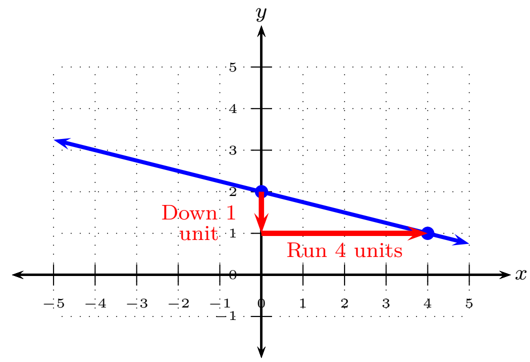
Example 2.6

Find the slope from the graph given of the line.



Solution.

We start at a well-defined point, preferably a point on the y-axis, i.e., the y-intercept. Then count the number of units we rise (up/down) and run (left/right) to reach the next well-defined point. We will start at $(0, 2)$ and reach the next point, $(4, 1)$. Notice we rise downward 1 units and run to the right 4 units. Using the ratio, $m = \frac{rise}{run}$, we get the rise to be -1 and the run to be 4:



$$m = \frac{rise}{run} = \frac{-1}{4}$$

Thus, the slope is $-\frac{1}{4}$.

Note

Looking at examples 2.5 and 2.6, notice the steepness. Since, in example 2.5, the slope was $\frac{3}{2}$, which is larger than the slope in 2.6, example 2.5 is a steeper line than example 2.6. Also, the negative in example 2.6 represents the line falling left to right; hence the negative slope.

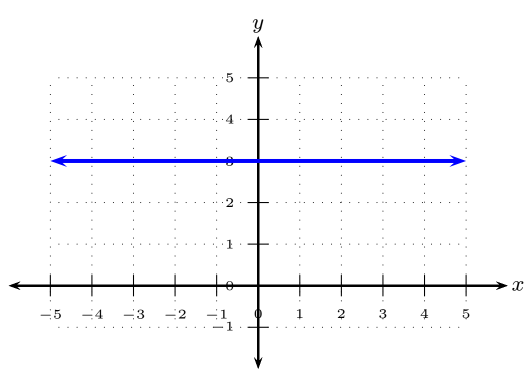
World Note

When French mathematicians *Rene Descartes* and *Pierre de Fermat* first developed the coordinate plane and the idea of graphing lines (and other functions), the y-axis was not a vertical line.

Let's look at two special cases with lines and their slope.

Example 2.7

Find the slope from the graph given of the line.

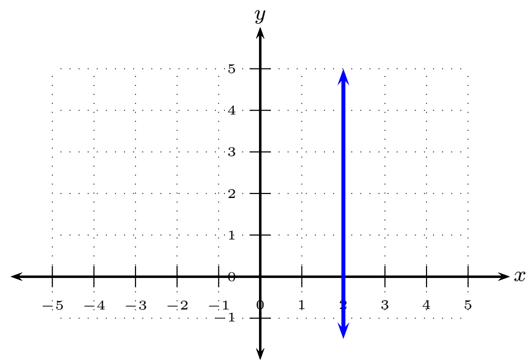


Solution.

In this graph, there is no rise, but the run is 3 units. The slope is $\frac{0}{3} = 0$. When the slope of a line is zero, then we know the line is a horizontal line and vise versa.

Example 2.8

Find the slope from the graph given of the line.



Solution.

In this graph, there is no run, but the rise is 2 units. The slope is $\frac{2}{0}$ = undefined. When the slope of a line is undefined, then we know the line is a vertical line and vise versa.

Note

As you can see there is a big difference between having a zero slope and having undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all. In fact, it is flat. Therefore, it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express the steepness. Hence, being an undefined slope.

2.1.3 Obtaining the slope of a line from two points

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y-values, we can calculate this by subtracting the y-values of a point. Similarly, if run is a change in the x-values, we can calculate this by subtracting the x-values of a point.

Definition

Slope, m, is the change of y divided by the change in x, i.e.,

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Example 2.9

Find the slope between the two points $(-4, 3)$ and $(2, -9)$.

Solution.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Substitute in the ordered-pairs}$$

$$m = \frac{-9 - 3}{2 - (-4)} \qquad \text{Simplify}$$

$$m = \frac{-12}{6} \qquad \text{Reduce}$$

$$m = -2 \qquad \text{Slope}$$

Since this slope is -2, then the graph of this line would be falling left to right.

Example 2.10

Find the slope between the two points $(4, 6)$ and $(2, -1)$.

Solution.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Substitute in the ordered-pairs}$$

$$m = \frac{-1 - 6}{2 - 4} \qquad \text{Simplify}$$

$$m = \frac{-7}{-2} \qquad \text{Reduce , dividing by } -1$$

$$m = \frac{7}{2} \qquad \text{Slope}$$

Since this slope is $\frac{7}{2}$, then the graph of this line would be rising left to right.

Example 2.11

Find the slope between the two points $(-4, -1)$ and $(-4, -5)$.

Solution.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Substitute in the ordered-pairs}$$

$$m = \frac{-5 - (-1)}{-4 - (-4)} \qquad \text{Simplify}$$

$$m = \frac{-4}{0} \qquad \text{Undefined}$$

$$m = \text{undefined} \qquad \text{Slope}$$

Since the slope is undefined, then the graph of this line is a vertical line.

Example 2.12

Find the slope between the two points $(3, 1)$ and $(-2, 1)$.

Solution.

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Substitute in the ordered-pairs} \\
m &= \frac{1 - 1}{-2 - 3} && \text{Simplify} \\
m &= \frac{0}{-5} && \text{Reduce} \\
m &= 0 && \text{Slope}
\end{aligned}
$$

Since the slope is zero, then the graph of this line is a horizontal line.

Example 2.13

Find the value of y between the points $(2, y)$ and $(5, -1)$ with slope -3.

Solution.

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{We will plug values into slope formula} \\
-3 &= \frac{-1 - y}{5 - 2} && \text{Simplify} \\
-3 &= \frac{-1 - y}{3} && \text{Multiply both sides by 3} \\
3 \cdot -3 &= \frac{-1 - y}{3} \cdot 3 && \text{Simplify} \\
-9 &= -1 - y && \text{Isolate the variable term} \\
-9 + 1 &= -1 - y + 1 && \text{Simplify} \\
-8 &= -y && \text{Multiply each side by } -1 \\
-1 \cdot -8 &= -y \cdot -1 && \\
8 &= y && \text{Value of } y
\end{aligned}
$$

Example 2.14

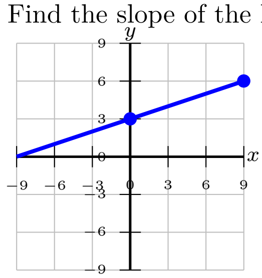
Find the value of x such that the slope between the points $(-3, 2)$ and $(x, 6)$ is $\frac{2}{5}$.

Solution.

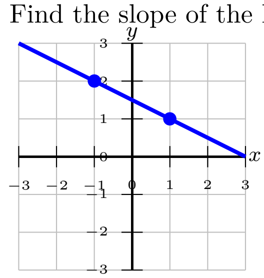
$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{We will plug values into slope formula}\\
\frac{2}{5} &= \frac{6-2}{x-(-3)} && \text{Simplify}\\
\frac{2}{5} &= \frac{4}{x+3} && \text{Multiply both sides by } x+3\\
(x+3)\cdot\frac{2}{5} &= \frac{4}{x+3}\cdot(x+3) && \text{Simplify}\\
\frac{2}{5}(x+3) &= 4 && \text{Distribute}\\
\frac{2}{5}x + \frac{6}{5} &= 4 && \text{Multiply by the LCD} = 5\\
5\cdot\frac{2}{5}x + 5\cdot\frac{6}{5} &= 4\cdot 5 && \text{Simplify}\\
2x + 6 &= 20 && \text{Isolate the variable term}\\
2x + 6 + (-6) &= 20 + (-6) && \text{Simplify}\\
2x &= 14 && \text{Multiply each side by the reciprocal of 2}\\
\frac{1}{2}\cdot 2x &= 14\cdot\frac{1}{2} && \text{Simplify}\\
x &= 7 && \text{Value of } x
\end{aligned}
$$

2.1.4 Graphing and Slope Homework

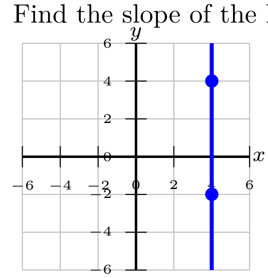
1. Find the slope of the line.



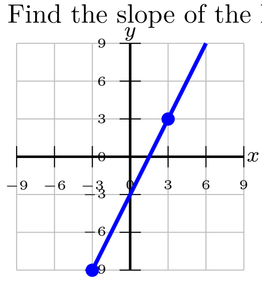
2. Find the slope of the line.



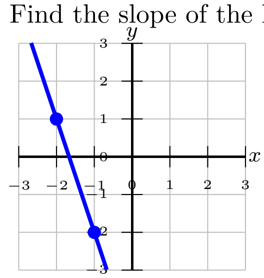
3. Find the slope of the line.



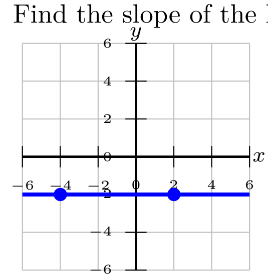
4. Find the slope of the line.



5. Find the slope of the line.



6. Find the slope of the line.



Find the slope of the line through each ordered-pair.

7. $(-2, 10), (-2, -15)$

8. $(-15, 10), (16, -7)$

9. $(10, 18), (-11, -10)$

10. $(-16, -14), (11, -14)$

11. $(-4, 14), (-16, 8)$

12. $(12, -19), (6, 14)$

13. $(-5, -10), (-5, 20)$

14. $(-17, 19), (10, -7)$

15. $(7, -14), (-8, -9)$

16. $(-5, 7), (-18, 14)$

17. $(1, 2), (-6, -14)$

18. $(13, -2), (7, 7)$

19. $(-3, 6), (-20, 13)$

20. $(13, 15), (2, 10)$

21. $(9, -6), (-7, -7)$

22. $(-16, 2), (15, -10)$

23. $(8, 11), (-3, -13)$

24. $(11, -2), (1, 17)$

25. $(-18, -5), (14, -3)$

26. $(19, 15), (5, 11)$

Find the value of x or y so that the line through the points has the given slope.

27. $(2, 6)$ and $(x, 2)$; $m = \frac{4}{7}$

28. $(-3, -2)$ and $(x, 6)$; $m = -\frac{8}{5}$

29. $(-8, y)$ and $(-1, 1)$; $m = \frac{6}{7}$

30. $(x, -7)$ and $(-9, -9)$; $m = \frac{2}{5}$

31. $(x, 5)$ and $(8, 0)$; $m = -\frac{5}{6}$

32. $(8, y)$ and $(-2, 4)$; $m = -\frac{1}{5}$

33. $(-2, y)$ and $(2, 4)$; $m = \frac{1}{4}$

34. $(x, -1)$ and $(-4, 6)$; $m = -\frac{7}{10}$

35. $(2, -5)$ and $(3, y)$; $m = 6$

36. $(6, 2)$ and $(x, 6)$; $m = -\frac{4}{5}$

2.2 Equations of lines

In this section, we discuss applying formulas to obtain equations of lines, graphing lines, and special cases. If we can identify some properties of the line, we may be able to graph the line much quicker and easier.

2.2.1 The slope-intercept formula

One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m, and the y-intercept can be represented by $(0, b)$, where b is the y-value at which the graph crosses the y-axis. Any other point on the line can be represented by (x, y).

Example 2.15

Prove the slope-intercept formula by using the slope formula for the line that has slope m, y-intercept $(0, b)$, and passes through the point (x, y).

Solution.

$$
\begin{aligned}
m &= \frac{y-b}{x-0} && \text{Simplify} \\
m &= \frac{y-b}{x} && \text{Multiply both sides by } x \\
mx &= y-b && \text{Add } b \text{ to both sides} \\
mx+b &= y && \\
y &= mx+b && \text{Slope-intercept formula}
\end{aligned}
$$

Slope-intercept formula

The ***slope-intercept formula*** of a linear equation is given by $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept.

Example 2.16

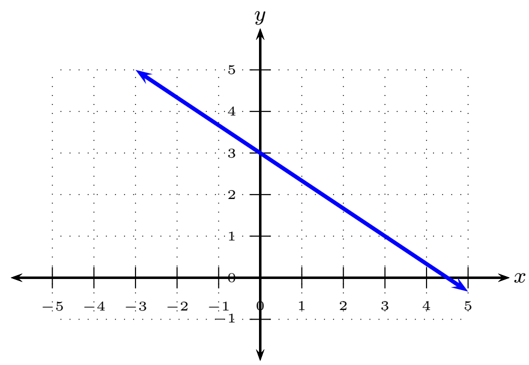
Find the equation of the line with slope $\frac{3}{4}$ and y-intercept $(0, -3)$.

Solution.

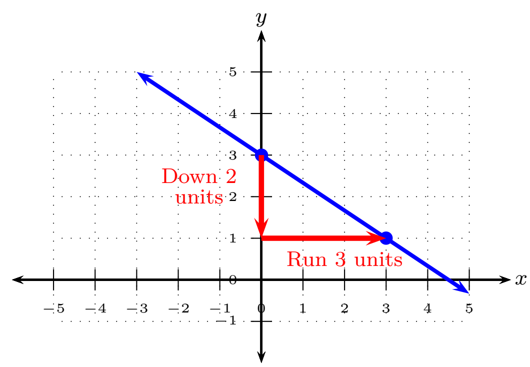
$$
\begin{aligned}
y &= mx+b && m \text{ is the slope}, b \text{ is the } y-\text{intercept} \\
y &= \frac{3}{4}x-3 && \text{Equation of the line}
\end{aligned}
$$

Example 2.17

Find the equation of the line.



Solution.



The y-intercept is where the graph crosses the y-axis. We can see, from the graph, that the line crosses the y-axis at $(0, 3)$. Hence, $b = 3$. To find the slope, we count the rise and run units. We can see that we rise downward 2 units and run to the right 3 units. Hence, the slope is $-\frac{2}{3}$. Now we can put the equation of the line together in slope-intercept form, where $m = -\frac{2}{3}$ and $b = 3$:

$$y = -\frac{2}{3}x + 3$$

2.2.2 Lines in slope-intercept form

We can also identify the slope and y-intercept, and graph the equation from a given equation. However, we need to be sure the equation is in slope-intercept form. If it is not, we will have to rewrite the equation in slope-intercept form, i.e., solve the equation for y. Then we can easily identify the slope and the y-intercept.

Example 2.18

Write the equation $2x - 4y = 6$ in slope-intercept form. Find the slope and y-intercept of the line.

Solution.

$$\begin{aligned} 2x - 4y &= 6 && \text{Isolate the variable term } -4y \\ 2x - 4y + (-2x) &= 6 + (-2x) && \text{Simplify} \\ -4y &= 6 - 2x && \text{Multiply by the reciprocal of } -4 \\ -\frac{1}{4} \cdot -4y &= -\frac{1}{4} \cdot 6 - 2x \cdot -\frac{1}{4} && \text{Simplify} \\ y &= \frac{1}{2}x - \frac{3}{2} && \text{Slope-intercept form} \end{aligned}$$

Next, we identify the slope and the y-intercept. If we line up the general slope-intercept form with the equation we obtained, we can easily see m and b:

$$\begin{array}{ccll} y & = & mx & +b \\ y & = & \frac{1}{2}x & -\frac{3}{2} \end{array}$$

Hence, $m = \frac{1}{2}$ and $b = -\frac{3}{2}$, i.e., the slope is $\frac{1}{2}$ and the y-intercept is $-\frac{3}{2}$.

2.2.3 Graphing lines

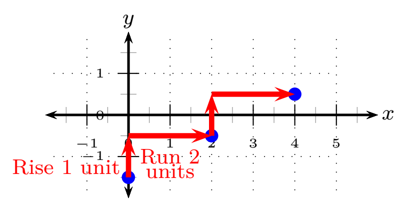
Once we have an equation in slope-intercept form, we can graph it by first plotting the y-intercept, then applying the slope to find a second point and even a third point. We connect these points to make a line. Let's look at example 2.18 and graph the line.

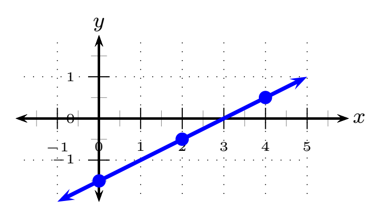
Example 2.19

Graph $y = \frac{1}{2}x - \frac{3}{2}$ by using the slope and y-intercept.

Solution.

The y-intercept, or b, is where the graph crosses the y-axis. We know from example 2.18 that the y-intercept is $-\frac{3}{2}$ and the line will cross the y-axis at $\left(0, -\frac{3}{2}\right)$. The slope is $\frac{1}{2}$, and, using $\frac{rise}{run}$, we need to rise upward 1 unit and run to the right 2 units to reach the next point. We continue the pattern to obtain a third point. Now we can connect the dots and create a well-defined line. Be sure to draw it to fill the grid.





Note

When we graphed a line by plotting points from a table, we obtained three points. Hence, in example 2.19, we obtained three points and then drew a line. Obtaining three points on a line is common practice and will help when drawing any line, even the special cases.

World Note

Before our current system of graphing, French mathematician, *Nicole Oresme*, in 1323, suggested graphing lines that would look more like bar graphs with a constant slope.

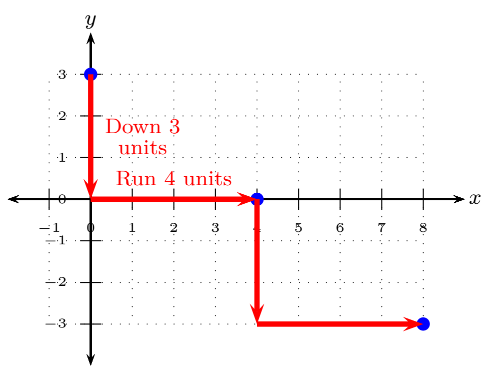
Example 2.20

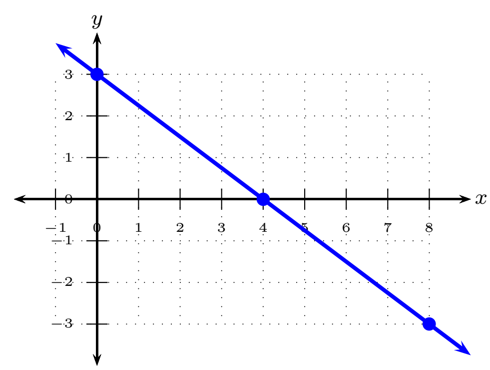
Write the equation $3x + 4y = 12$ in slope-intercept form. Find the slope and y-intercept of the line and then graph the line.

Solution.

$$\begin{aligned}
3x + 4y &= 12 && \text{Isolate the variable term } 4y\\
3x + 4y + (-3x) &= 12 + (-3x) && \text{Simplify}\\
4y &= 12 - 3x && \text{Multiply by the reciprocal of 4}\\
\frac{1}{4}\cdot 4y &= \frac{1}{4}\cdot 12 - 3x \cdot \frac{1}{4} && \text{Simplify}\\
y &= -\frac{3}{4}x + 3 && \text{Slope-intercept form}
\end{aligned}$$

No we can graph the line. We see the y-intercept is 3 and the line will cross the y-axis at $(0, 3)$. The slope is $-\frac{3}{4}$, and, using $\frac{rise}{run}$, we need to rise downward 3 units and run to the right 4 units to reach the next point. We continue the pattern to obtain a third point. Now we can connect the dots and create a well-defined line. Be sure to draw it to fill the grid.





2.2.4 Vertical and horizontal lines

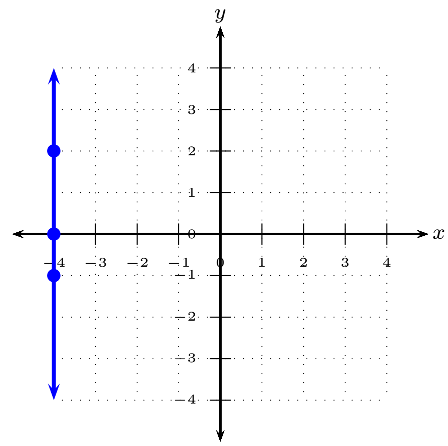
Lines with zero or undefined slope can make a problem seem very different. Zero slope, or a horizontal line, will simply have a slope of zero. So, the equation simply becomes $y = b$ or y equal to the y-coordinate of the graph. If we have undefined slope, or a vertical line, the equation can't be written in slope-intercept form because the slope is undefined. Hence, there is no y in these equations. We will simply make x equal to the x-coordinate of the graph.

Example 2.21

Graph the line $x = -4$.

Solution.

Since $x = -4$ is a vertical line, then we know this line has no slope and the line is at its steepest. Every x-coordinate on this line is -4 and the line has no run. We can graph this line easily by plotting three points where the x-coordinate is -4. Let's plot $(-4,-1)$, $(-4,0)$, and $(-4,2)$; then connect the points with a well-defined line.

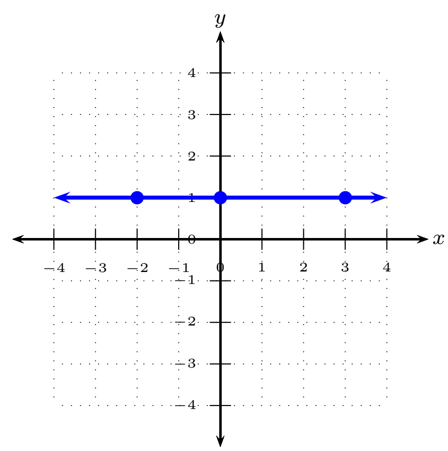


Example 2.22

Graph the line $y = 1$.

Solution.

Since $y = 1$ is a horizontal line, then we know this line has zero slope and the line is at its flattest. Every y-coordinate on this line is 1 and the line has no rise. We can graph this line easily by plotting three points where the y-coordinate is 1. Let's plot $(3,1)$, $(0,1)$, and $(-2,1)$; then connect the points with a well-defined line.



2.2.5 Point-slope formula

The slope-intercept form has the advantage of being simple to remember and use. However, it has one major disadvantage: we must know the y-intercept in order to graph the line. Generally, we do not know the y-intercept, but, usually, know one or more points on the line that are not the y-intercept. In these cases, we can't use the slope-intercept equation, so we will need a more general formula to assist us in graphing lines. If the slope of a line is m, and point (x_1, y_1) be a particular point on the line, and any other point on the line be (x, y), then we can use this to find this general formula.

$$
\begin{aligned}
m, (x_1, y_1), (x, y) & \qquad \text{Recall slope formula} \\
\frac{y_2 - y_1}{x_2 - x_1} = m & \qquad \text{Plug in values} \\
\frac{y - y_1}{x - x_1} = m & \qquad \text{Multiply both sides by } (x - x_1) \\
y - y_1 = m(x - x_1) & \qquad \text{New formula}
\end{aligned}
$$

Point-slope formula

The ***point-slope formula*** is given by

$$y - y_1 = m\big(x - x_1\big),$$

given the slope m and point $\big(x_1, y_1\big)$ is on the line.

Example 2.23

Using the point-slope formula, write the equation of the line passing through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

Solution.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Plug values into point-slope formula} \\ y - (-4) &= \frac{3}{5}(x - 3) && \text{Simplify signs} \\ y + 4 &= \frac{3}{5}(x - 3) && \text{Equation in point-slope form} \end{aligned}$$

Note

Often, we will prefer final answers be written in slope-intercept form. If the directions prefer the equations of the line in slope-intercept form, we can distribute the slope, then solve for y.

Example 2.24

Let's rewrite example 2.23 in slope-intercept form: $y + 4 = \frac{3}{5}(x - 3)$

Solution.

$$\begin{aligned} y + 4 &= \frac{3}{5}(x - 3) && \text{Distribute} \\ y + 4 &= \frac{3}{5}x - \frac{9}{5} && \text{Isolate the variable term } y \\ y + 4 + (-4) &= \frac{3}{5}x - \frac{9}{5} + (-4) && \text{Simplify} \\ y &= \frac{3}{5}x - \frac{11}{5} && \text{Slope-intercept form} \end{aligned}$$

Example 2.25

Write the equation of the line passing through the point $(-6, 2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

Solution.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Substitute values into the point-slope formula} \\
y - 2 &= -\frac{2}{3}(x - (-6)) && \text{Simplify} \\
y - 2 &= -\frac{2}{3}(x + 6) && \text{Distribute} \\
y - 2 &= -\frac{2}{3}x - 4 && \text{Isolate the variable term } y \\
y - 2 + 2 &= -\frac{2}{3}x - 4 + 2 && \text{Simplify} \\
y &= -\frac{2}{3}x - 2 && \text{Slope-intercept form}
\end{aligned}
$$

2.2.6 Obtaining a line given two points

In order to find the equation of a line, we need to know the slope. If we aren't given the slope, but only two points on the line, then we complete some preliminary work to obtain the slope. Then we can use the point-slope formula as usual to obtain the equation of the line.

Example 2.26

Find the equation of the line passing through the points $(-3, 4)$ and $(-1, -2)$ in slope-intercept form.

Solution.

Since we are given two points, we can use the slope formula to obtain the slope:

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Substitute in the ordered-pairs} \\
m &= \frac{-2 - 4}{-1 - (-3)} && \text{Simplify} \\
m &= \frac{-6}{2} \\
m &= -3 && \text{Slope}
\end{aligned}
$$

Now that we have the slope, we can plug-n-chug the slope and one of the points into the point-slope formula. Notice we have two points and we can choose either one; the results will be the same. Let's choose $(-3, 4)$ with the slope $m = -3$.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope} \\
y - 4 &= -3(x - (-3)) && \text{Simplify} \\
y - 4 &= -3(x + 3) && \text{Distribute} \\
y - 4 &= -3x - 9 && \text{Isolate the variable term } y \\
y - 4 + 4 &= -3x - 9 + 4 && \text{Simplify} \\
y &= -3x - 5 && \text{Slope-intercept form}
\end{aligned}
$$

Example 2.27

Find the equation of the line through the points $(6, -2)$ and $(-4, 1)$ in slope-intercept form.

Solution.

Since we are given two points, we can use the slope formula to obtain the slope:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Substitute in the ordered-pairs} \\ m &= \frac{1-(-2)}{-4-6} && \text{Simplify} \\ m &= -\frac{3}{10} && \text{Slope} \end{aligned}$$

Now that we have the slope, we can plug-n-chug the slope and one of the points into the point-slope formula. Notice we have two points and we can choose either one; the results will be the same. Let's choose $(-4, 1)$ with the slope $m = -\frac{3}{10}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope} \\ y - 1 &= -\frac{3}{10}(x - (-4)) && \text{Simplify} \\ y - 1 &= -\frac{3}{10}(x + 4) && \text{Distribute} \\ y - 1 &= -\frac{3}{10}x - \frac{6}{5} && \text{Isolate the variable term } y \\ y - 1 + 1 &= -\frac{3}{10}x - \frac{6}{5} + 1 && \text{Simplify} \\ y &= -\frac{3}{10}x - \frac{1}{5} && \text{Slope-intercept form} \end{aligned}$$

World Note

The city of *Konigsberg* (now Kaliningrad, Russia) had a river that flowed through the city breaking it into several parts. There were 7 bridges that connected the parts of the city. In 1735, Leonhard Euler considered the question of whether it was possible to cross each bridge exactly once and only once. It turned out that this problem was impossible, but the work laid the foundation of what would later become graph theory.

2.2.7 Equations of Lines Homework

Write the equation of the line in slope-intercept form given the slope and the y-intercept.

1. $m = 2$, y-intercept $= 5$

2. $m = 1$, y-intercept $= -4$

3. $m = -\frac{3}{4}$, y-intercept $= -1$

4. $m = \frac{1}{3}$, y-intercept $= 1$

5. $m = -6$, y-intercept $= 4$

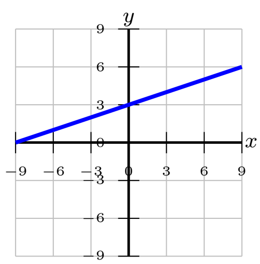
6. $m = -1$, y-intercept $= -2$

7. $m = -\frac{1}{4}$, y-intercept $= 3$

8. $m = \frac{2}{5}$, y-intercept $= 5$

Write the equation of the line in slope-intercept form given the graph or equation.

9.



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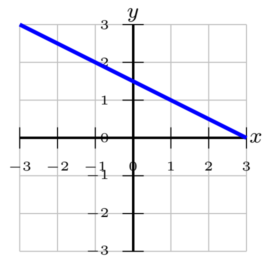

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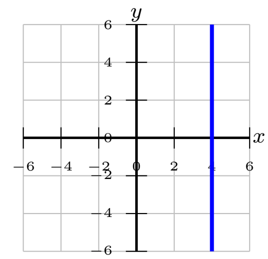

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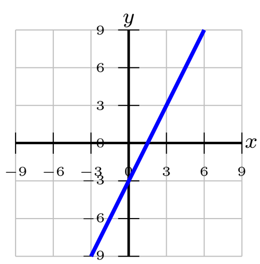

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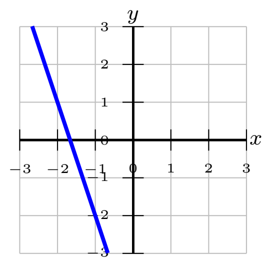

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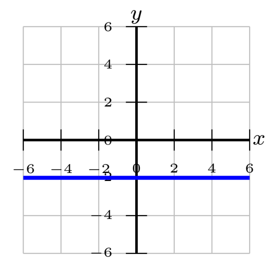


15. $x + 10y = -37$

16. $2x + y = -1$

17. $7x - 3y = 24$

18. $x = -8$

19. $y - 4 = -(x + 5)$

20. $y - 4 = 4(x - 1)$

21. $y + 5 = -4(x - 2)$

22. $y + 1 = -\frac{1}{2}(x - 4)$

23. $x - 10y = 3$

24. $6x - 11y = -70$

25. $4x + 7y = 28$

26. $x - 7y = -42$

27. $y - 5 = \frac{5}{2}(x - 2)$

28. $y - 3 = -\frac{2}{3}(x + 3)$

29. $0 = x - 4$

30. $y + 2 = \frac{6}{5}(x + 5)$

Sketch the graph of each line.

31. $y = \frac{1}{3}x + 4$ **32.** $y = \frac{6}{5}x - 5$ **33.** $y = \frac{3}{2}x$

34. $x - y + 3 = 0$ **35.** $-y - 4 + 3x = 0$ **36.** $-3y = -5x + 9$

37. $y = -\frac{1}{5}x - 4$ **38.** $y = -\frac{3}{2}x - 1$ **39.** $y = -\frac{3}{4}x + 1$

40. $4x + 5 = 5y$ **41.** $-8 = 6x - 2y$ **42.** $-3y = 3 - \frac{3}{2}x$

Write the equation of the line in point-slope form given a point passing through the line and its slope.

43. $(2, 3)$; $m =$ undefined **44.** $(2, 2)$; $m = \frac{1}{2}$ **45.** $(-1, -5)$; $m = 9$

46. $(-4, 1)$; $m = \frac{3}{4}$ **47.** $(0, -2)$; $m = -3$ **48.** $(0, -5)$; $m = -\frac{1}{4}$

49. $(-5, -3)$; $m = \frac{1}{5}$ **50.** $(-1, 4)$; $m = -\frac{5}{4}$ **51.** $(1, 2)$; $m = 0$

52. $(2, 1)$; $m = -\frac{1}{2}$ **53.** $(2, -2)$; $m = -2$ **54.** $(4, -3)$; $m = -2$

55. $(-1, 1)$; $m = 4$ **56.** $(0, 2)$; $m = -\frac{5}{4}$ **57.** $(-1, -4)$; $m = -\frac{2}{3}$

Write the equation of the line in slope-intercept form given a point passing through the line and its slope.

58. $(-1, -5)$; $m = 2$ **59.** $(5, -1)$; $m = -\frac{3}{5}$ **60.** $(-4, 1)$; $m = \frac{1}{2}$

61. $(4, -2)$; $m = -\frac{3}{2}$ **62.** $(-5, -3)$; $m = -\frac{2}{5}$ **63.** $(2, -2)$; $m = 1$

64. $(-3, 4)$, $m =$undefined **65.** $(-4, 2)$; $m = -\frac{1}{2}$ **66.** $(2, -2)$; $m = -2$

67. $(-2, -2)$; $m = -\frac{2}{3}$ **68.** $(4, -3)$; $m = -\frac{7}{4}$ **69.** $(-2, 0)$; $m = -\frac{5}{2}$

70. $(3, 3)$; $m = \frac{7}{3}$ **71.** $(-4, -3)$ $m = 0$ **72.** $(-2, -5)$; $m = 2$

Write the equation of the line in point-slope form given two points on the line.

73. $(-4, 3)$ and $(-3, 1)$ **74.** $(5, 1)$ and $(-3, 0)$ **75.** $(-4, -2)$ and $(0, 4)$

76. $(3, 5)$ and $(-5, 3)$ **77.** $(3, -3)$ and $(-4, 5)$ **78.** $(1, 3)$ and $(-3, 3)$

79. $(-4, 5)$ and $(4, 4)$ **80.** $(-4, 1)$ and $(4, 4)$ **81.** $(-1, -4)$ and $(-5, 0)$

Write the equation of the line in slope-intercept form given two points on the line.

82. $(-5, 1)$ and $(-1, -2)$

83. $(-5, 5)$ and $(2, -3)$

84. $(4, 1)$ and $(1, 4)$

85. $(0, 2)$ and $(5, -3)$

86. $(0, 3)$ and $(-1, -1)$

87. $(-5, -1)$ and $(5, -2)$

88. $(1, -1)$ and $(-5, -4)$

89. $(0, 1)$ and $(-3, 0)$

90. $(0, 2)$ and $(2, 4)$

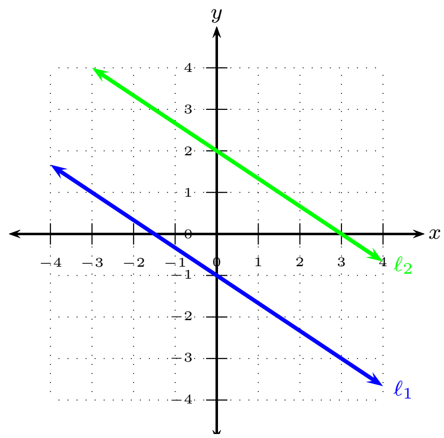
2.3 Parallel and perpendicular lines

In this section, we discuss parallel and perpendicular lines. The relationship between parallel lines and between perpendicular lines is unique, where the slope will be most interesting to us in this section.

2.3.1 The slope of parallel and perpendicular lines

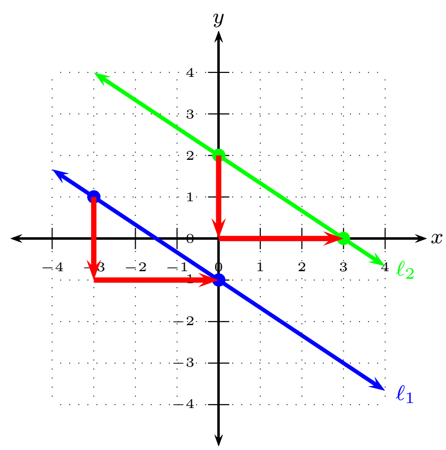
Example 2.28

Find the slope of each line and compare. What is interesting about the slopes?



Solution.

Looking at ℓ_1, we can start at $(-3,1)$ and reach the next point at $(0,-1)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_1's slope is $-\dfrac{2}{3}$. Now let's look at ℓ_2 and obtain its slope. We will start at $(0,2)$ and reach the next point at $(3,0)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_2's slope is $-\dfrac{2}{3}$. The slopes of ℓ_1 and ℓ_2 are $-\dfrac{2}{3}$; they have the same exact slope but different y-intercepts.

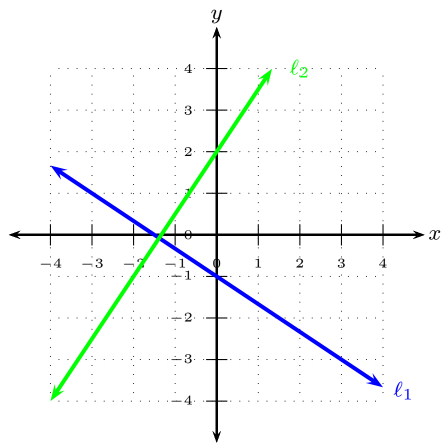


Definition

Let m_1 and m_2 be slopes for lines ℓ_1 and ℓ_2, respectively. Lines ℓ_1 and ℓ_2 are ***parallel*** to each other if they have the same slope, but different y-intercepts, i.e., $m_1 = m_2$.

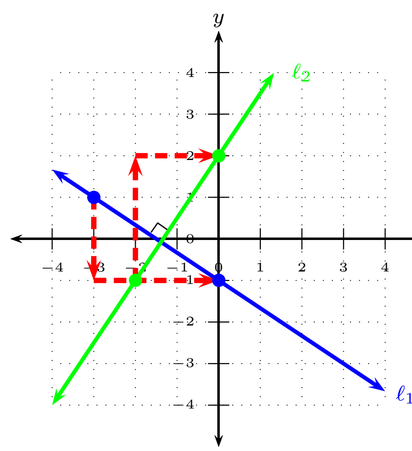
Example 2.29

Find the slope of each line and compare. What is interesting about the slopes?



Solution.

Looking at ℓ_1, we can start at $(-3, 1)$ and reach the next point at $(0, -1)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_1's slope is $-\frac{2}{3}$. Now let's look at ℓ_2 and obtain its slope. We will start at $(-2, -1)$ and reach the next point at $(0, 2)$. We see that we will move up three units and run to the right 2 units. Hence, ℓ_2's slope is $\frac{3}{2}$. The slopes of ℓ_1 and ℓ_2 are negative reciprocals, i.e., if one has slope m, then a line perpendicular to it will have slope $-\frac{1}{m}$. Also, note that if two lines are perpendicular, they create a right angle at the intersection.



Definition

Let m_1 and m_2 be slopes for lines ℓ_1 and ℓ_2, respectively. Lines ℓ_1 and ℓ_2 are ***perpendicular*** to each other if they have negative reciprocal slopes, i.e., ℓ_1 has slope m_1 and ℓ_2 has slope $m_2 = -\frac{1}{m_1}$.

Example 2.30

Find the slope of a line parallel to $5y - 2x = 7$.

Solution.

We need to rewrite the equation in slope-intercept form. Then we can identify the slope and the slope for a line parallel to it.

$$\begin{aligned}
5y - 2x &= 7 && \text{Isolate the variable term } 5y \\
5y - 2x + 2x &= 7 + 2x && \text{Simplify} \\
5y &= 2x + 7 && \text{Multiply by the reciprocal of } 5 \\
\frac{1}{5} \cdot 5y &= \frac{1}{5} \cdot 2x + 7 \cdot \frac{1}{5} && \text{Simplify} \\
y &= \frac{2}{5}x + \frac{7}{5}
\end{aligned}$$

We see the slope of the given line is $\frac{2}{5}$. By the definition, a line parallel will have the same slope $\frac{2}{5}$.

Example 2.31

Find the slope of a line perpendicular to $3x - 4y = 2$

Solution.

We need to rewrite the equation in slope-intercept form. Then we can identify the slope and the slope for a line perpendicular to it.

$$\begin{aligned}
3x - 4y &= 2 && \text{Isolate the variable term } -4y \\
3x - 4y + (-3x) &= 2 + (-3x) && \text{Simplify} \\
-4y &= -3x + 2 && \text{Multiply by the reciprocal of } -4 \\
-\frac{1}{4} \cdot -4y &= -\frac{1}{4} \cdot -3x + 2 \cdot -\frac{1}{4} && \text{Simplify} \\
y &= \frac{3}{4}x - \frac{1}{2}
\end{aligned}$$

We see the slope of the given line is $\frac{3}{4}$. By the definition, a line perpendicular will have a negative reciprocal slope $-\frac{4}{3}$.

2.3.2 Obtain equations for parallel and perpendicular lines

Once we have obtained the slope for a line perpendicular or parallel, it is possible to find the complete equation of the second line if we are given a point on the second line.

Example 2.32

Find the equation of a line passing through $(4, -5)$ and parallel to $2x - 3y = 6$.

Solution.

First, we can rewrite the given line in slope-intercept form to obtain the slope for a line parallel to it:

$$\begin{aligned}
2x - 3y &= 6 && \text{Isolate the variable term } -3y\\
2x - 3y + (-2x) &= 6 + (-2x) && \text{Simplify}\\
-3y &= -2x + 6 && \text{Multiply by the reciprocal of } -3\\
-\frac{1}{3} \cdot -3y &= -\frac{1}{3} \cdot -2x + 6 \cdot -\frac{1}{3} && \text{Simplify}\\
y &= \frac{2}{3}x - 2
\end{aligned}$$

We see the slope of the given line is $\frac{2}{3}$. By the definition, a line parallel will have the same slope $\frac{2}{3}$. Next, we can use the point-slope formula to obtain the equation of the line passing through $(4, -5)$ with slope $\frac{2}{3}$:

$$\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope}\\
y - (-5) &= \frac{2}{3}(x - 4) && \text{Simplify signs}\\
y + 5 &= \frac{2}{3}(x - 4) && \text{A line parallel to } 2x - 3y = 6 \text{ in point-slope form}
\end{aligned}$$

Example 2.33

Find the equation of the line, in slope-intercept form, passing through $(6, -9)$ and perpendicular to $y = -\frac{3}{5}x + 4$.

Solution.

Since the given line is in slope-intercept form, we can easily observe the slope and the slope for a line perpendicular. We see the slope of the given line is $-\frac{3}{5}$. By the definition, a line perpendicular will have a negative reciprocal slope $\frac{5}{3}$. Next, we can use the point-slope formula to obtain the equation, in slope-intercept form, of the line passing through $(6, -9)$ with slope $\frac{5}{3}$:

$$\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope}\\
y - (-9) &= \frac{5}{3}(x - 6) && \text{Simplify signs}\\
y + 9 &= \frac{5}{3}(x - 6) && \text{Distribute}\\
y + 9 &= \frac{5}{3}x - 10 && \text{Isolate the variable term } y\\
y + 9 + (-9) &= \frac{5}{3}x - 10 + (-9) && \text{Simplify}\\
y &= \frac{5}{3}x - 19 && \text{A line perpendicular to } y = -\frac{3}{5}x + 4 \text{ in slope-intercept form}
\end{aligned}$$

✎ Note

Lines with zero slopes and undefined slopes may seem like opposites because a horizontal line has slope zero and a vertical line has slope that is undefined. Since a horizontal line is perpendicular to a vertical line, we can say, by definition, the slopes are negative reciprocals, i.e., $m_1 = 0$ would imply $m_2 = -\frac{1}{0}$, which is undefined.

Example 2.34

Find the equation of the line passing through $(3, 4)$ and perpendicular to $x = -2$.

Solution.

Since $x = -2$ is a vertical line, then this line has slope that is undefined. Hence, a line perpendicular to it will have slope zero, i.e., $m = 0$. Next, we can use the point-slope formula to obtain the equation, in slope-intercept form, of the line passing through $(3, 4)$ with slope $m = 0$:

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope} \\ y - 4 &= 0(x - 3) && \text{Distribute} \\ y - 4 &= 0 && \text{Isolate the variable term } y \\ y - 4 + 4 &= 0 + 4 && \text{Simplify} \\ y &= 4 && \text{A line perpendicular to } x = -2 \end{aligned}$$

Now, since we are aware that a line perpendicular to a vertical line is a horizontal line and we were given a point $(3, 4)$, we could have easily jumped to the equation, $y = 4$.

2.3.3 Parallel and Perpendicular Lines Homework

Given the line, find the slope of a line parallel.

1. $y = 2x + 4$

2. $y = 4x - 5$

3. $x - y = 4$

4. $7x + y = -2$

5. $y = -\frac{2}{3}x + 5$

6. $y = -\frac{10}{3}x - 5$

7. $6x - 5y = 20$

8. $3x + 4y = -8$

Given the line, find the slope of a line perpendicular.

9. $x = 3$

10. $y = -\frac{1}{3}x$

11. $x - 3y = -6$

12. $x + 2y = 8$

13. $y = -\frac{1}{2}x - 1$

14. $y = \frac{4}{5}x$

15. $3x - y = -3$

16. $8x - 3y = -9$

Find the equation of the line, in point-slope form, passing through the point and given the line to be parallel or perpendicular.

17. $(2, 5)$; parallel to $x = 0$

18. $(5, 2)$; parallel to $y = \frac{7}{5}x + 4$

19. $(3, 4)$; parallel to $y = \frac{9}{2}x - 5$

20. $(1, -1)$; parallel to $y = -\frac{3}{4}x + 3$

21. $(2, 3)$; parallel to $y = \frac{7}{5}x + 4$

22. $(-1, 3)$; parallel to $y = -3x - 1$

23. $(4, 2)$; parallel to $x = 0$

24. $(1, 4)$; parallel to $y = \frac{7}{5}x + 2$

25. $(1, -5)$; perpendicular to $-x + y = 1$

26. $(1, -2)$; perpendicular to $-x + 2y = 2$

27. $(5, 2)$; perpendicular to $5x + y = -3$

28. $(1, 3)$; perpendicular to $-x + y = 1$

29. $(4, 2)$; perpendicular to $-4x + y = 0$

30. $(-3, -5)$; perpendicular to $3x + 7y = 0$

31. $(2, -2)$; perpendicular to $3y - x = 0$

32. $(-2, 5)$; perpendicular to $y - 2x = 0$

Find the equation of the line, in slope-intercept form, passing through the point and given the line to be parallel or perpendicular.

33. $(4, -3)$; parallel to $y = -2x$

34. $(-5, 2)$; parallel to $y = \frac{3}{5}x$

35. $(-3, 1)$; parallel to $y = -\frac{4}{3}x - 1$

36. $(-4, 0)$; parallel to $y = -\frac{5}{4}x + 4$

37. $(-4, -1)$; parallel to $y = -\frac{1}{2}x + 1$

38. $(2, 3)$; parallel to $y = \frac{5}{2}x - 1$

39. $(-2, -1)$; parallel to $y = -\frac{1}{2}x - 2$

40. $(-5, -4)$; parallel to $y = \frac{3}{5}x - 2$

41. $(4, 3)$; perpendicular to $x + y = -1$

42. $(-3, -5)$; perpendicular to $x + 2y = -4$

43. $(5, 2)$; perpendicular to $x = 0$

44. $(5, -1)$; perpendicular to $-5x + 2y = 10$

45. $(-2, 5)$; perpendicular to $-x + y = -2$

46. $(2, -3)$; perpendicular to $-2x + 5y = -10$

47. $(4, -3)$; perpendicular to $-x + 2y = -6$

48. $(-4, 1)$; perpendicular to $4x + 3y = -9$

2.4 Graphing Linear Equations: Answers to the Homework Exercises

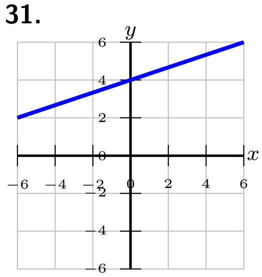
Graphing and Slope

1. $\frac{1}{3}$
3. undefined
5. -3
7. undefined
9. $\frac{4}{3}$
11. $\frac{1}{2}$
13. undefined
15. $-\frac{1}{3}$
17. $\frac{16}{7}$
19. $-\frac{7}{17}$
21. $\frac{1}{16}$
23. $\frac{24}{11}$
25. $\frac{1}{16}$
27. $x = -5$
29. $y = -5$
31. $x = \frac{23}{6}$
33. $y = -12$
35. $y = -\frac{29}{6}$

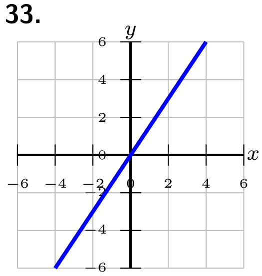
Equations of Lines

1. $y = 2x + 5$
3. $y = -\frac{3}{4}x - 1$
5. $y = -6x + 4$
7. $y = -\frac{1}{4}x + 3$
9. $y = \frac{1}{3}x + 3$
11. $x = 4$
13. $y = -3x + 5$
15. $y = -\frac{1}{10}x - \frac{37}{10}$
17. $y = \frac{7x}{3} - 8$
19. $y = 4x$
21. $y = -4x + 3$
23. $y = \frac{1}{10}x - \frac{3}{10}$
25. $y = -\frac{4}{7}x + 4$
27. $y = \frac{5}{2}x$
29. N/A

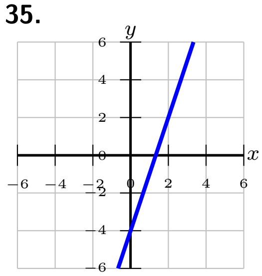
31.



33.



35.



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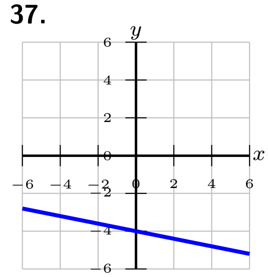

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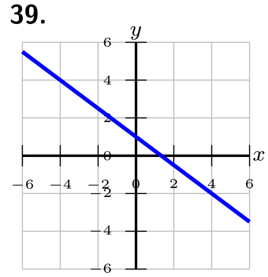
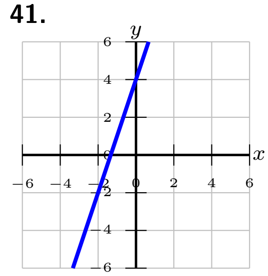


41.



43. $x - 2 = 0$
45. $y - (-5) = 9(x - (-1))$

47. $y-(-2)=-3(x-0)$

49. $y-(-3)=\frac{1}{5}(x-(-5))$

51. $y-2=0(x-1)$

53. $y-(-2)=-2(x-2)$

55. $y-1=4(x-(-1))$

57. $y-(-4)=-\frac{2}{3}(x-(-1))$

59. $y=-\frac{3}{5}x+2$

61. $y=-\frac{3}{2}x+4$

63. $y=x-4$

65. $y=-\frac{1}{2}x$

67. $y=-\frac{2}{3}x-\frac{10}{3}$

69. $y=-\frac{5}{2}x-5$

71. $y=-3$

73. $y-3=-2(x+4)$

75. $y+2=\frac{3}{2}(x+4)$

77. $y+3=-\frac{8}{7}(x-3)$

79. $y-5=-\frac{1}{8}(x+4)$

81. $y+4=-(x+1)$

83. $y=-\frac{8}{7}x-\frac{5}{7}$

85. $y=-x+2$

87. $y=-\frac{1}{10}x-\frac{3}{2}$

89. $y=\frac{1}{3}x+1$

Parallel and Perpendicular Lines

1. $m_{||}=2$

3. $m_{||}=1$

5. $m_{||}=-\frac{2}{3}$

7. $m_{||}=\frac{6}{5}$

9. $m_{\perp}=0$

11. $m_{\perp}=-3$

13. $m_{\perp}=2$

15. $m_{\perp}=-\frac{1}{3}$

17. $y=5$

19. $y-4=\frac{9}{2}(x-3)$

21. $y-3=\frac{7}{5}(x-2)$

23. $x=4$

25. $y+5=-(x-1)$

27. $y-2=\frac{1}{5}(x-5)$

29. $y-2=-\frac{1}{4}(x-4)$

31. $y+2=-3(x-2)$

33. $y=-2x+5$

35. $y=-\frac{4}{3}x-3$

37. $y=-\frac{1}{2}x-3$

39. $y=-\frac{1}{2}x-2$

41. $y=x-1$

43. $y=2$

45. $y=-x+3$

47. $y=-2x+5$

Chapter 3

Linear Inequalities in One and Two Variables

Chapter Objectives

By the end of this chapter, the student should be able to

- Solve linear inequalities in one variable including compound and absolute value inequalities
- Graph the solutions to linear inequalities in one and two variables
- Write solutions to linear inequalities in one variable in interval notation

3.1 Linear inequalities

When there is a solution to an equation such as $x = 4$, this solution is unique and is the only solution that makes the statement true. However, with *inequalities*, the solution is an interval of numbers in which make the inequality true.

Definition

An ***inequality*** is a relation between two numbers or two sets of numbers (or elements) in which are not strictly equal, but greater than (or equal to) or less than (or equal to).

$$\begin{array}{ll} > & \text{greater than} \\ \geq & \text{greater than or equal to} \\ < & \text{less than} \\ \leq & \text{less than or equal to} \end{array}$$

World Note

English mathematician *Thomas Harriot* first used the inequality symbols in 1631. However, they were not immediately accepted, where symbols $\sqsubset$ and $\sqsupset$ were already coined by another English mathematician, *William Oughtred*.

3.1.1 Graphing linear inequalities

If we have an statement such as $x < 4$, this means a solution can be any number smaller than 4 such as $-2, 0, 3, 3.9$ or even 3.999999999 as long as it is smaller than 4. If we have a statement such as $x \geq -2$, this means a solution can be any number greater than or equal to -2, such as $5, 0, -1, -1.9999$, or even -2. Because we don't have one value as the solution, it is often useful to draw a picture of the solutions to the inequality on a number line.

Definition

A ***linear inequality*** in one variable is an inequality of the form

$$ax + b < c,$$

where the inequality is written in the same form for $>, \leq, \geq$.

Interval notation

We rewrite the $>, <, \leq, \geq$ symbols as parenthesis and brackets, i.e., (,),], [, respectively, when we write the inequality in interval notation.

Case 1. If $x < a$, then the equivalent set of numbers in interval notation is $(-\infty, a)$; similarly, If $x \leq a$, then the equivalent set of numbers in interval notation is $(-\infty, a]$.

Case 2. If $x > a$, then the equivalent set of numbers in interval notation is (a, ∞); similarly, If $x \geq a$, then the equivalent set of numbers in interval notation is $[a, \infty)$.

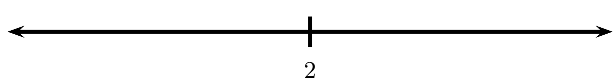
Example 3.1

Graph the inequality and rewrite the inequality in interval notation: $x < 2$

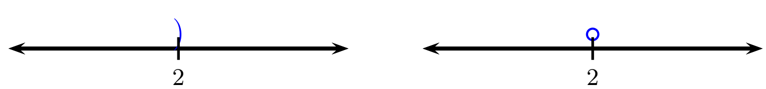
Solution.

We will complete this example in steps and use this method for the remaining future examples involving inequalities.

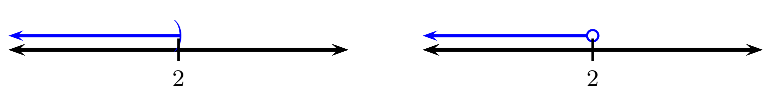
Step 1. Draw a number line and mark the number in the inequality on the line.



Step 2. Write a parenthesis or bracket, depending on the inequality sign, over the number on the number line. Note, we can easily use $\bullet$ for $\leq$ and $\geq$, and $\circ$ for $<$ and $>$. Since we have $x < 2$, then we use $\circ$ or) on the number line:



Step 3. Draw a line connecting the] or $\circ$ in the direction where lies the solution. Since $x < 2$, then we want all numbers less than 2; hence, we draw the line to the left to represent all numbers less than 2:



Note: Even though there are two graphs, one will suffice. At the discretion of the instructor, brackets and parenthesis, or closed and open circles will be used when graphing inequalities.

Step 4. Rewrite the inequality in interval notation. Since $x < 2$, we can see this is case 1 and so the interval is $(-\infty, 2)$. We use a parenthesis on the 2 since it is $<$.

✎ Note

In example 3.1, we use a parenthesis on $-\infty$ because $\pm\infty$ are not real numbers and symbolizes "some large (positive or negative) number beyond any real number." It is common practice to always use parentheses on infinities for intervals.

World Note

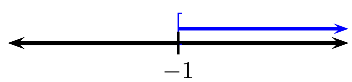
The symbol for infinity was first used by the Romans even though, at the time, the number was used for 1,000. The Greeks also used the symbol for 10,000.

Example 3.2

Graph the inequality and rewrite the inequality in interval notation: $y \geq -1$

Solution.

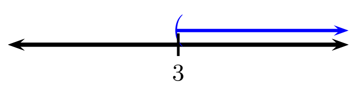
We start by labeling the number line with -1. Then draw a line to the right since all numbers greater than (or equal to) -1 are to the right:



Next, we write $y \geq -1$ in interval notation. This is case 2 and the interval is $[-1, \infty)$; we use a bracket because of $\geq$.

Example 3.3

Given the graph, write the equivalent inequality and interval notation:

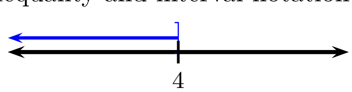


Solution.

Since the graph shows all numbers (strictly) greater than 3, then the inequality is $x > 3$; we use $>$ because of the (on the number line. The equivalent interval would be, using case 2, $(3, \infty)$.

Example 3.4

Given the graph, write the equivalent inequality and interval notation:



Solution.

Since the graph shows all numbers less than (or equal to) 4, then the inequality is $x \leq 4$; we use $\leq$ because of the] on the number line. The equivalent interval would be, using case 1, $(-\infty, 4]$.

3.1.2 Solving linear inequalities

Properties of inequalities

There are two properties are used with inequalities:

Property 1. *Addition Property of Inequalities*: For numbers a, b, c,

$$a < b \text{ is equivalent to } a + c < b + c,$$

where the form is the same for $>, \leq, \geq$. I.e., you can add a number to one side of the inequality as long as you add the same number to the other side. (This should remind you of the addition property for equations.)

Property 2. *Multiplication Property of Inequalities*: For numbers a, b, c and $c \neq 0$,

$$a < b \text{ is equivalent to } a \cdot c < b \cdot c,$$

where the form is the same for $>, \leq, \geq$. I.e., you can multiply a nonzero number to one side of the inequality as long as you multiply the same nonzero number to the other side.

However, if $c < 0$, then the inequality reverses, i.e, ***if you multiply or divide by a negative, then reverse the inequality symbol***.

Example 3.5

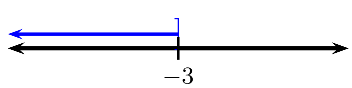
Solve the inequality. Graph the solution and write the solution in interval notation.

$$5 - 2x \geq 11$$

Solution.

$$\begin{aligned} 5 - 2x &\geq 11 && \text{Add the opposite of 5 to each side} \\ 5 - 2x + (-5) &\geq 11 + (-5) && \text{Simplify} \\ -2x &\geq 6 && \text{Multiply by the reciprocal of } -2 \\ \left(-\frac{1}{2}\right) \cdot -2x &\leq 6 \cdot \left(-\frac{1}{2}\right) && \text{Simplify and reverse the inequality sign} \\ x &\leq -3 && \text{Solution} \end{aligned}$$

Let's graph the solution $x \leq -3$:



Next, we write $x \leq -3$ in interval notation. This is case 1 and the interval is $(-\infty, -3]$; we use a bracket because of $\leq$.

Example 3.6

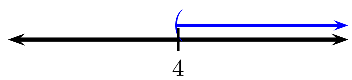
Solve the inequality. Graph the solution and write the solution in interval notation.

$$3(2x - 4) + 4x < 4(3x - 7) + 8$$

Solution.

$$
\begin{aligned}
3(2x-4)+4x &< 4(3x-7)+8 && \text{Distribute}\\
6x-12+4x &< 12x-28+8 && \text{Combine like terms}\\
10x-12 &< 12x-20 && \text{Isolate the variable term}\\
10x-12+(-10x) &< 12x-20+(-10x) && \text{Simplify}\\
-12 &< 2x-20 && \text{Add the opposite of } -20 \text{ to each side}\\
-12+20 &< 2x-20+20 && \text{Simplify}\\
8 &< 2x && \text{Multiply by the reciprocal of 2}\\
\frac{1}{2}\cdot 8 &< 2x\cdot\frac{1}{2} && \text{Simplify}\\
4 &< x && \text{Rewrite the inequality with } x \text{ on the left side}\\
x &> 4 && \text{Solution}
\end{aligned}
$$

Let's graph the solution $x > 4$:



Next, we write $x > 4$ in interval notation. This is case 2 and the interval is $(4, \infty)$; we use a parenthesis because of $>$.

Warning!

It is important to be careful when the solution to the inequality has the isolated variable on the right side like in example 3.6, i.e., $4 < x$ rather than $x > 4$. It is best practice to write the variable on the left side after isolating the variable. This will minimize confusion when graphing the solution and writing the solution in interval notation.

3.1.3 Tripartite inequalities

A special type of compound inequality is called a *tripartite inequality*, when the variable (or expression containing the variable) is between two numbers. When solving these types of inequalities with three parts, we will apply the properties of inequalities to all three parts (rather than two sides) to isolate the variable in the middle.

Interval notation for tripartite inequalities

Case 1. If $a < x < b$, then the equivalent set of numbers in interval notation is (a, b); similarly, If $a \leq x \leq b$, then the equivalent set of numbers in interval notation is $[a, b]$.

Case 2. If $a < x \leq b$, then the equivalent set of numbers in interval notation is $(a, b]$; similarly, If $a \leq x < b$, then the equivalent set of numbers in interval notation is $[a, b)$.

Example 3.7

Solve the inequality. Graph the solution and write the solution in interval notation.

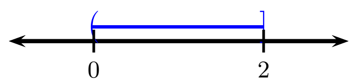
$$-6 \leq -4x + 2 < 2$$

Solution.

Solve the inequality, graph the solution, and give interval notation.

$$\begin{array}{rl}
-6 \leq -4x + 2 < 2 & \text{Add the opposite of 2 to each part of the inequality} \\
-6 + (-2) \leq -4x + 2 + (-2) < 2 + (-2) & \text{Simplify} \\
-8 \leq -4x < 0 & \text{Multiply by the reciprocal of } -4 \\
-\frac{1}{4} \cdot -8 \geq -\frac{1}{4} \cdot -4x > -\frac{1}{4} \cdot 0 & \text{Simplify} \\
2 \geq x > 0 & \text{Rewrite with } \geq, > \text{ signs} \\
0 < x \leq 2 & \text{Solution}
\end{array}$$

Let's graph the solution $0 < x \leq 2$:



Next, we write $0 < x \leq 2$ in interval notation. This is case 2 and the interval is $(0, 2]$.

3.1.4 Linear Inequalities Homework

Graph the inequalities and rewrite the inequalities in interval notation.

1. $n > -5$

2. $-2 \geq k$

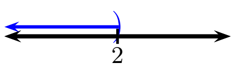
3. $5 \geq x$

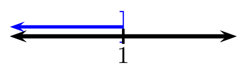
4. $n > 4$

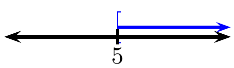
5. $1 \geq k$

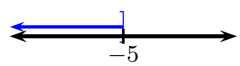
6. $-5 < x$

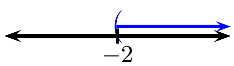
Given the graph, write the equivalent inequality and interval notation.

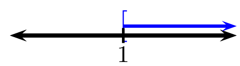
7. 

8. 

9. 

10. 

11. 

12. 

Solve the inequality. Graph the solution and write the solution in interval notation.

13. $\dfrac{x}{11} \geq 10$

14. $2 + r < 3$

15. $8 + \dfrac{n}{3} \geq 6$

16. $2 > \dfrac{a-2}{5}$

17. $-47 \geq 8 - 5x$

18. $-2(3 + k) < -44$

19. $18 < -2(-8 + p)$

20. $24 \geq -6(m - 6)$

21. $-r - 5(r - 6) < -18$

22. $24 + 4b < 4(1 + 6b)$

23. $-5v - 5 < -5(4v + 1)$

24. $4 + 2(a + 5) < -2(-a - 4)$

25. $-(k - 2) > -k - 20$

26. $-2 \leq \dfrac{n}{13}$

27. $\dfrac{m}{5} \leq -\dfrac{6}{5}$

28. $11 > 8 + \dfrac{x}{2}$

29. $\dfrac{v-9}{-4} \leq 2$

30. $\dfrac{6+x}{12} \leq -1$

31. $-7n - 10 \geq 60$

32. $5 \geq \dfrac{x}{5} + 1$

33. $-8(n - 5) \geq 0$

34. $-60 \geq -4(-6x - 3)$

35. $-8(2 - 2n) \geq -16 + n$

36. $-36 + 6x > -8(x + 2) + 4x$

37. $3(n + 3) + 7(8 - 8n) < 5n + 5 + 2$

38. $-(4 - 5p) + 3 \geq -2(8 - 5p)$

39. $3 \leq 9 + x \leq 7$

40. $11 < 8 + k \leq 12$

41. $-3 < x - 1 < 1$

42. $-4 < 8 - 3m \leq 11$

43. $-16 \leq 2n - 10 \leq -22$

44. $0 \geq \dfrac{x}{9} \geq -1$

45. $-11 \leq n - 9 \leq -5$

46. $1 \leq \dfrac{p}{8} \leq 0$

3.2 Compound inequalities

Several inequalities are combined together to form ***compound inequalities***. There are three types of compound inequalities which we will investigate in this section: *and*, *or*, and *tripartite*.

Let A and B be sets.

- The ***union*** of two sets contains all the elements contained in either set (or both sets). The ***union*** is denoted $A \cup B$.
- The ***intersection*** of two sets contains only the elements that are in both sets. The ***intersection*** is denoted $A \cap B$.

3.2.1 Compound inequalites with *or*

For *or* compound inequalities, the solution is a true statement from either one inequality, the other inequality, or both. This should remind you of the *union* of two sets except, in this case, we are going to determine the solution by graphing each inequality, and write the solution in interval notation. *Hint*: We will use the union symbol, $\cup$, in the solution.

Example 3.8

Solve the compound inequality. Graph the solution and write the solution in interval notation.

$$2x - 5 > 3 \quad \text{or} \quad 4 - x \geq 6$$

Solution.

We begin the solution by solving for each inequality.

$$\begin{aligned} 2x - 5 > 3 \quad &\text{or} \quad 4 - x \geqslant 6 && \text{Solve each inequality} \\ 2x > 8 \quad &\text{or} \quad -x \geqslant 2 && \text{Divide by the coefficient of } x \\ x > 4 \quad &\text{or} \quad x \leqslant -2 && \text{Solution in inequality notation} \end{aligned}$$

Let's graph each of these inequalities to determine the union of the two sets.

Looking for the union of these two sets, we see the solution is all the numbers to the left of -2 (inclusive), or to the right of 4, or in both. Hence, in interval notation, the solution is

$$(-\infty, -2] \cup (4, \infty)$$

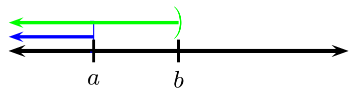
World Note

The symbol for infinity was first used by the Romans, although, at the time, the number was used for 1,000. The Greeks also used the symbol for 10,000.

Note

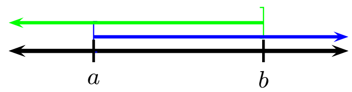
There are special cases with the *or* compound inequalities when obtaining a solution.

Case 1. As one graph in contained in the other, the union of the two sets will be the larger set.



Hence, in this case, the solution is $(-\infty, b)$.

Case 2. As the graph has a nonempty intersection, where the entire number line is included, then the solution is *all real numbers*.



Hence, in this case, the solution is $(-\infty, \infty)$.

3.2.2 Compound inequalities with *and*

For *and* compound inequalities, the solution is a true statement from *both* inequalities. This should remind you of the *intersection* of two sets except, in this case, we are going to determine the solution by graphing each inequality, and write the solution in interval notation. *Hint*: The solution is one interval, and no set symbols are used with *and* problems.

Example 3.9

Solve the compound inequality. Graph the solution and write the solution in interval notation.

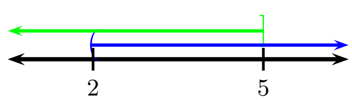
$$2x + 8 \geq 5x - 7 \quad \text{and} \quad 5x - 3 > 3x + 1$$

Solution.

We begin the solution by solving for each inequality.

$$\begin{aligned} 2x + 8 \geq 5x - 7 \quad &\text{and} \quad 5x - 3 > 3x + 1 && \text{Solve each inequality} \\ 8 \geq 3x - 7 \quad &\text{and} \quad 2x - 3 > 1 && \text{Isolate the variable term} \\ 15 \geq 3x \quad &\text{and} \quad 2x > 4 && \text{Divide by the coefficient of } x \\ 5 \geq x \quad &\text{and} \quad x > 2 && \text{Solution in inequality notation} \end{aligned}$$

Let's graph each of these inequalities to determine the intersection of the two sets.

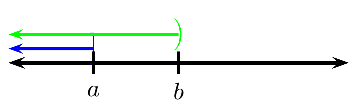


Looking for the intersection of these two sets, we see the solution is all the numbers in between 2 and 5 (inclusive). Hence, in interval notation, the solution is $(2, 5]$.

✎ Note

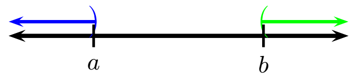
There are special cases with the *and* compound inequalities when obtaining a solution.

Case 1. As one graph in contained in the other, the intersection of the two sets will be the smaller set.



Hence, in this case, the solution is $(-\infty, a)$.

Case 2. As the graphs do not intersect, where there is a gap between sets, then there is no solution, i.e., $\varnothing$.



Hence, in this case, the solution is no solution or $\varnothing$.

Recall, 3.1.3 from the previous section. *Tripartite* inequalities are *and* compound inequalities, where we are find the intersection of the two sets. Notice, the solutions in 3.1.3 were one interval as with *and* compound inequalities.

Example 3.10

Solve the compound inequality. Graph the solution and write the solution in interval notation.

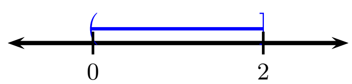
$$-6 \leq -4x + 2 < 2$$

Solution.

We begin the solution by applying the properties of inequalities to every side of the inequality.

$-6 \leq -4x + 2 < 2$	Subtract 2 from all three parts
$-8 \leq -4x < 0$	Divide all three parts by -4
$2 \geq x > 0$	Flip entire statement so values increase from left to right
$0 < x \leq 2$	Solution in inequality notation

Let's graph this inequality to determine the intersection of the two sets.



We can see the intersection is all the numbers in between 0 and 2 (inclusive). Hence, in interval notation, the solution is $(0, 2]$.

3.2.3 Compound Inequalities Homework

Solve the compound inequalities. Graph the solution and write the solution in interval notation.

1. $\frac{n}{3} \leq -3 \quad \text{or} \quad -5n \leq -10$

2. $x + 7 \geq 12 \quad \text{or} \quad 9x < -45$

3. $x - 6 < -13 \quad \text{or} \quad 6x \leq -60$

4. $\frac{v}{8} > -1 \quad \text{and} \quad v - 2 < 1$

5. $-8 + b < -3 \quad \text{and} \quad 4b < 20$

6. $a + 10 \geq 3 \quad \text{and} \quad 8a \leq 48$

7. $3 \leq 9 + x \leq 7$

8. $11 < 8 + k \leq 12$

9. $-3 < x - 1 < 1$

10. $-4 < 8 - 3m \leq 11$

11. $-16 \leq 2n - 10 \leq -22$

12. $-5b + 10 \leq 30 \quad \text{and} \quad 7b + 2 \leq -40$

13. $3x - 9 < 2x + 10 \quad \text{and} \quad 5 + 7x \leq 10x - 10$

14. $-8 - 6v \leq 8 - 8v \quad \text{and} \quad 7v + 9 \leq 6 + 10v$

15. $1 + 5k \leq 7k - 3 \quad \text{or} \quad k - 10 > 2k + 10$

16. $2x + 9 \geq 10x + 1 \quad \text{and} \quad 3x - 2 < 7x + 2$

17. $6m \geq -24 \quad \text{or} \quad m - 7 < -12$

18. $10r > 0 \quad \text{or} \quad r - 5 < -12$

19. $9 + n < 2 \quad \text{or} \quad 5n > 40$

20. $-9x < 63 \quad \text{and} \quad \frac{x}{4} < 1$

21. $-6n \leq 12 \quad \text{and} \quad \frac{n}{3} \leq 2$

22. $-6 + v \geq 0 \quad \text{and} \quad 2v > 4$

23. $0 \geq \frac{x}{9} \geq -1$

24. $-11 \leq n - 9 \leq -5$

25. $1 \leq \frac{p}{8} \leq 0$

26. $3 + 7r > 59 \quad \text{or} \quad -6r - 3 > 33$

27. $-6 - 8x \geq -6 \quad \text{or} \quad 2 + 10x > 82$

28. $n + 10 \geq 15 \quad \text{or} \quad 4n - 5 < -1$

29. $4n + 8 < 3n - 6 \quad \text{or} \quad 10n - 8 \geq 9 + 9n$

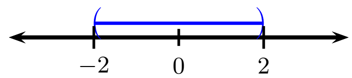
30. $5 - 2a \geq 2a + 1 \quad \text{or} \quad 10a - 10 \geq 9a + 9$

31. $8 - 10r \leq 8 + 4r \quad \text{or} \quad -6 + 8r < 2 + 8r$

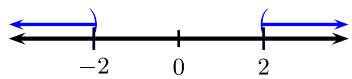
32. $-9m + 2 < -10 - 6m \quad \text{or} \quad -m + 5 \geq 10 + 4m$

3.3 Absolute value inequalities

When an inequality contains absolute value, we need to rewrite the inequality without the absolute value in order to solve the inequality. Let's consider $|x| < 2$. Recall, absolute value is defined as the distance from zero. The idea behind solving $|x| < 2$ is to find all numbers that have a distance from zero that is less than 2. Let's take a look at this graphically.



This graph should remind us of *tripartite (and) compound inequalities* and it is! Now let's consider $|x| > 2$. The idea behind solving $|x| > 2$ is to find all numbers that have a distance from zero that is more than 2. Let's take a look at this graphically.



This graph should remind us of *or compound inequalities* and it is! Absolute inequalities will remind us of the concepts from compound inequalities.

Definition

Given below are the cases for absolute value for inequalities in one variable, where a is a real number.

Case 1. If $|x| < a$, then $-a < x < a$.

Case 2. If $|x| > a$, then $x < -a$ or $x > a$.

Similar cases for $\leq$ and $\geq$, respectively.

World Note

The phrase "absolute value" comes from German mathematician *Karl Weierstrass* in 1876, though he used the absolute value symbol for complex numbers. The first known use of the symbol for integers comes from a 1939 edition of a college algebra textbook.

Example 3.11

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

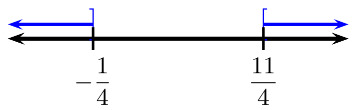
$$|4x - 5| \geq 6$$

Solution.

We begin the solution by rewriting the absolute value inequality using the cases in the definition.

$$\begin{aligned}
|4x - 5| &\geq 6 && \text{Case 2. Rewrite as two inequalities using } \textit{or} \\
4x - 5 \geq 6 \quad \text{or} \quad 4x - 5 &\leq -6 && \text{Solve each inequality} \\
4x \geq 11 \quad \text{or} \quad 4x &\leq -1 && \text{Divide by the coefficient of } x \\
x \geq \frac{11}{4} \quad \text{or} \quad x &\leq -\frac{1}{4} && \text{Solution in inequality notation}
\end{aligned}$$

Let's graph each of these inequalities to determine the union of the two sets.



Looking for the union of these two sets, we see the solution is all the numbers to the left of $-\frac{1}{4}$ (inclusive), or to the right of $\frac{11}{4}$ (inclusive), or in both. Hence, in interval notation, the solution is

$$\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{11}{4}, \infty\right)$$

3.3.1 Isolate the absolute value

Recall. In the previous chapter, when solving absolute value equations, we were required to isolate the absolute value term before applying the definition. In this section, we will do the same method because *methods never change, only problems.*

Example 3.12

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

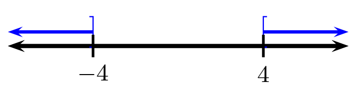
$$-4 - 3|x| \leq -16$$

Solution.

We begin the solution by rewriting the absolute value inequality where the absolute value term is isolated on the left side. Then we can apply the cases in the definition.

$-4 - 3\lvert x\rvert \leq -16$	Isolate the absolute value term
$-3\lvert x\rvert \leq -12$	Divide both sides by -3
$\lvert x\rvert \geq 4$	Case 2. Rewrite as two inequalities using *or*
$x \geq 4 \quad \text{or} \quad x \leq -4$	Solution in inequality notation

Let's graph each of these inequalities to determine the union of the two sets.



Looking for the union of these two sets, we see the solution is all the numbers to the left of -4 (inclusive), or to the right of 4 (inclusive), or in both. Hence, in interval notation, the solution is

$$(-\infty, -4] \cup [4, \infty)$$

Example 3.13

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

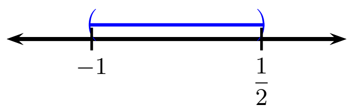
$$9 - 2|4x + 1| > 3$$

Solution.

We begin the solution by rewriting the absolute value inequality where the absolute value term is isolated on the left side. Then we can apply the cases in the definition.

$$
\begin{aligned}
9-2|4x+1| &> 3 && \text{Isolate the absolute value term}\\
-2|4x+1| &> -6 && \text{Divide both sides by } -2\\
|4x+1| &< 3 && \text{Case 1. Rewrite as a tripartite inequality}\\
-3 < 4x+1 &< 3 && \text{Solve, start by subtracting 1 from all three parts}\\
-4 < 4x &< 2 && \text{Divide by the coefficient of } x\\
-1 < x &< \frac{1}{2} && \text{Solution in inequality notation}
\end{aligned}
$$

Let's graph this inequality to determine the intersection of the two sets.



We can see the intersection is all the numbers in between -1 and $\frac{1}{2}$. Hence, in interval notation, the solution is $\left(-1, \frac{1}{2}\right)$.

3.3.2 Special cases with absolute value inequalities

There are special cases with the absolute value inequalities when obtaining a solution. We recall that after evaluating the absolute value of an expression, the result is positive. Hence, if we were to isolate the absolute value on the left side of the inequality, then that result is always positive, i.e., $|x| \geq 0$, no matter the case.

Example 3.14

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

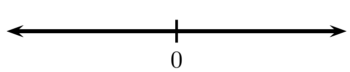
$$12 + 4|6x-1| < 4$$

Solution.

We begin the solution by rewriting the absolute value inequality where the absolute value term is isolated on the left side. Then we can apply the cases in the definition.

$$
\begin{aligned}
12+4|6x-1| &< 4 && \text{Isolate the absolute value term}\\
4|6x-1| &< -8 && \text{Divide by the coefficient 4}\\
|6x-1| &< -2 && \text{The absolute value is always non-negative}
\end{aligned}
$$

By definition of absolute value, $|6x-1| \geq 0$. Hence, $|6x-1|$ could never be less than zero, let alone less than -2. Thus, this inequality has no solution or $\varnothing$. To graph a $\varnothing$ solution, we leave a number line empty:



Note

Let's say, in example 3.14 we didn't recognize that $|6x-1|$ was less than a negative number and we solved as usual. After applying Case 1. in the definition, we would obtain a false statement. In example 3.14, we would have obtained

$$2 < 6x - 1 < -2,$$

but 2 isn't less than -2 ($2 \not< -2$). Hence, a false statement. At this point, we would see that the inequality $|6x-1| < -2$ has no solution.

Example 3.15

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

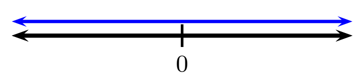
$$5 - 6|x+7| \leq 17$$

Solution.

We begin the solution by rewriting the absolute value inequality where the absolute value term is isolated on the left side. Then we can apply the cases in the definition.

$$\begin{aligned} 5 - 6|x+7| &\leq 17 && \text{Isolate the absolute value term} \\ -6|x+7| &\leq 12 && \text{Divide by the coefficient } -6 \\ |x+7| &\geq -2 && \text{The absolute value is always non-negative} \end{aligned}$$

By definition of absolute value, $|x+7| \geq 0$. Hence, $|x+7|$ is always greater than -2. Thus, this inequality has every real number as a solution and the solution is all real numbers or $(-\infty, \infty)$. To graph a $(-\infty, \infty)$ solution, we draw a line including the entire number line:



Note

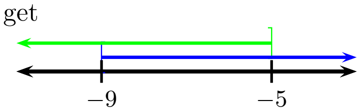
Let's say, in example 3.15 we didn't recognize that $|x+7|$ was always greater than a negative number and we solved as usual. After applying Case 2. in the definition, we would obtain the same result. In example 3.15, we would have obtained

$$x + 7 \geq -2 \quad \text{or} \quad x + 7 \leq 2,$$

which results in, after solving,

$$x \geq -9 \quad \text{or} \quad x \leq -5$$

Graphing each inequality, we get



Hence, the solution is $(-\infty, \infty)$.

3.3.3 Absolute Value Inequalities Homework

Solve the absolute value inequalities. Graph the solution and write the solution in interval notation.

1. $|x| < 3$

2. $|2x| < 6$

3. $|x-2| < 6$

4. $|x-7| < 3$

5. $|3x-2| < 9$

6. $1+2|x-1| \leq 9$

7. $6-|2x-5| \geq 3$

8. $|3x| > 5$

9. $|x-3| \geq 3$

10. $|3x-5| > 3$

11. $4+3|x-1| \geq 10$

12. $3-2|x-5| \leq -15$

13. $-2-3|4-2x| \geq -8$

14. $4-5|-2x-7| < -1$

15. $3-2|4x-5| \geq 1$

16. $-5-2|3x-6| < -8$

17. $4-4|-2x+6| > -4$

18. $|-10+x| \geq 8$

19. $|x| \leq 8$

20. $|x+3| < 4$

21. $|x-8| < 12$

22. $|x+3| \leq 4$

23. $|2x+5| < 9$

24. $10-3|x-2| \geq 4$

25. $|x| > 5$

26. $|x-4| > 5$

27. $|2x-4| > 6$

28. $3-|2-x| < 1$

29. $3-2|3x-1| \geq -7$

30. $4-6|-6-3x| \leq -5$

31. $-3-2|4x-5| \geq 1$

32. $-2+3|5-x| \leq 4$

33. $-2-3|-3x-5 \geq -5$

34. $6-3|1-4x| < -3$

35. $-3-4|-2x-5| \geq -7$

3.4 Linear inequalities in two variables

Previously, we graphed inequalities in one variable, but now we learn to graph inequalities in two variables. Although this section may seem similar to linear equations in two variables, linear inequalities in two variables have many applications. For example, business owners want to know when revenue is greater than cost so that their business makes a profit, e.g., revenue > cost.

Definition

A ***linear inequality*** in two variables is an inequality of the form

$$ax + by < c,$$

where the inequality is written in the same form for $>, \leq, \geq$ and $a, b \neq 0$.

Recall. The solution to a linear inequality in one variable is an interval of numbers, e.g., $(-\infty, \infty)$, $[-2, 3)$, $(1, 9)$, $[-7, -3]$, etc.

3.4.1 Verifying solutions

Solution to a linear inequality in two variables

An ordered pair (x, y) is a solution to a linear inequality in two variables, $ax + by < c$, if the ordered pair (x, y) makes the inequality true, where the the same is for $>, \leq, \geq$ and $a, b \neq 0$.

Example 3.16

Verify whether each ordered pair is a solution to the inequality $y > x + 4$.

a) $(0, 0)$ **b)** $(1, 6)$

Solution.

We substitute the ordered pairs into the inequality and determine if the results are true.

a) Let's substitute $(0, 0)$ into the inequality and determine if the left side is greater than the right side.

$$\begin{aligned} y &\overset{?}{>} x + 4 && \text{Substitute } x = 0 \text{ and } y = 0 \\ 0 &\overset{?}{>} 0 + 4 && \text{Simplify} \\ 0 &\not> 4 && \text{✗ False} \end{aligned}$$

Hence, $(0, 0)$ is not a solution to the inequality $y > x + 4$.

b) Let's substitute $(1, 6)$ into the inequality and determine if the left side is greater than the right side.

$$\begin{aligned} y &\overset{?}{>} x + 4 && \text{Substitute } x = 1 \text{ and } y = 6 \\ 6 &\overset{?}{>} 1 + 4 && \text{Simplify} \\ 6 &> 5 && \checkmark \text{ True} \end{aligned}$$

Hence, $(1, 6)$ is a solution to the inequality $y > x + 4$.

3.4.2 Boundary lines

If we are given a linear inequality, $ax + by < c$, we could see from example 3.16 that not all ordered pairs are a solution, only some. Why? Well, notice that $(0, 0)$ is below the line $y = x + 4$ and $(1, 6)$ is above the line $y = x + 4$. This implies that ordered pairs in certain regions are solutions to the inequality $y > x + 4$. Hence, the line $y = x + 4$ is critical when finding solutions to the inequality. We call the line $y = x + 4$ a ***boundary line***, a line that separates the ordered pairs that are solutions and the ordered pairs that are not solutions of the linear inequality in two variables $y > x + 4$.

Definition

A linear equation in two variables $ax + by = c$ is called the ***boundary line***, the line that separates the region where $ax + by > c$ and from the region where $ax + by < c$.

Note

Since there are four inequality symbols: $>, <, \geq, \leq$, then we have linear inequalities in two variables that include the boundary, e.g., inequalities with $\leq$ and $\geq$, and linear inequalities in two variables that exclude the boundary, e.g., inequalities with $<$ and $>$.

We can use the table below to help identify the boundary line, determine whether to include the boundary line, and the way the boundary line looks graphically.

Case 1.	**Case 2.**
$ax + by < c$	$ax + by \leq c$
$ax + by > c$	$ax + by \geq c$
Boundary line: $ax + by = c$	Boundary line: $ax + by = c$
Boundary line is excluded in solution	Boundary line is included in solution
Boundary line is dashed	Boundary line is solid

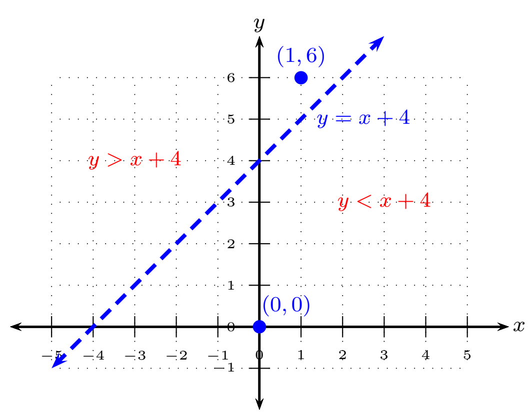
Example 3.17

Let's revisit example 3.16 and graph the boundary line and points $(0, 0)$ and $(1, 6)$.

Solution.

Since we have $y > x + 4$, we can see from the table above we have Case 1. and the boundary line is excluded. We represent this by graphing the line $y = x + 4$ as a dashed line.

We can see from the graph that the point $(0, 0)$ lies below the boundary line $y = x + 4$ and point $(1, 6)$ lies above the boundary line. Recall, point $(1, 6)$ was verified as a solution of $y > x + 4$ in example 3.16. Furthermore, any ordered pair that lies above $y = x + 4$ will verify as a solution, i.e., making the inequality true. We usually represent this area by *shading the region* where the set of ordered pairs make the inequality true.



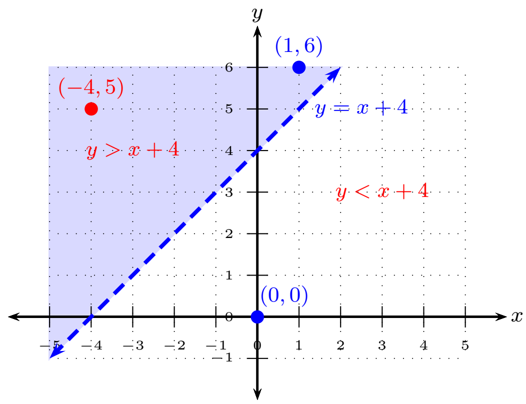
3.4.3 Graphing linear inequalities

Example 3.18

Graph the inequality from example 3.16.

Solution.

Since we know that $(1, 6)$ is a solution to the inequality, then we shade above the dashed boundary line:



We see that any ordered pair in the shaded region is a solution to the inequality. For example, let's pick $(-4, 5)$ and verify this is a solution:

$$\begin{aligned} y &\stackrel{?}{>} x + 4 && \text{Substitute } x = -4 \text{ and } y = 5 \\ 5 &\stackrel{?}{>} -4 + 4 && \text{Simplify} \\ 5 &> 0 && \checkmark \text{ True} \end{aligned}$$

Hence, $(-4, 5)$ is a solution to the inequality $y > x + 4$.

Steps for graphing linear inequalities in two variables

Given a linear inequality in two variables, $ax + by < c$, we use the steps below to graph $ax + by < c$, where the the same process is applied for $>, \leq, \geq$ and $a, b \neq 0$.

Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

Step 2. Graph the boundary line according to the two cases:

Case 1. If the inequality is $<$ or $>$, then the boundary line is dashed.

Case 2. If the inequality is $\geq$ or $\leq$, then the boundary line is solid.

Step 3. Select a test point that is *not on the boundary line.* **Ask:** *Does this ordered pair make the inequality true?*

Step 4. If the ordered pair is

- a solution to the inequality, i.e., makes the inequality true, then shade the side that includes the ordered pair.
- not a solution, then shade the opposite side of the boundary line.

☠ Warning!

If we choose a test point on the boundary line, we will obtain an identity, where both sides of the inequality symbol are the same number. Hence, it is critical to choose a point not on the boundary line.

Example 3.19

Graph the inequality: $2x - y > 3$.

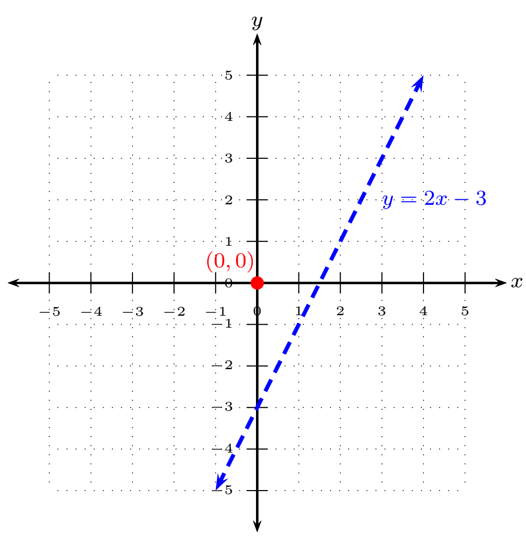
Solution.

Let's follow the steps given above to graph the inequality.

Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

$$\begin{aligned} 2x - y &> 3 \\ -y &> -2x + 3 \\ y &< 2x - 3 \end{aligned}$$

Step 2. Graph the boundary line according to the two cases. Since the given inequality is $<$, then we have Case 1.



Step 3. Select a test point that is *not on the boundary line.* **Ask:** *Does this ordered pair make the inequality true?*

Let's pick the test point $(0,0)$ as it is a great choice!

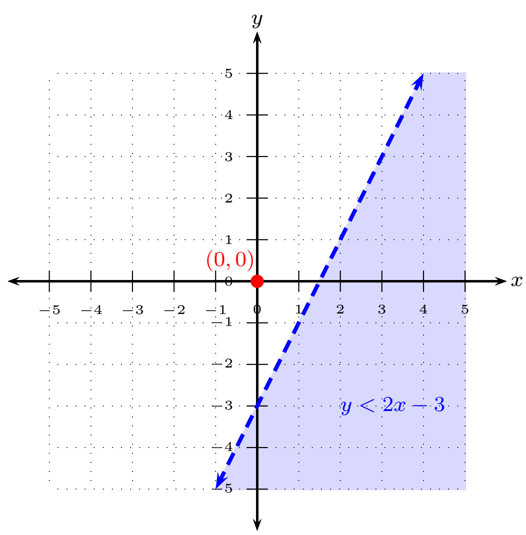
$$\begin{aligned} y &\stackrel{?}{<} 2x-3 \\ 0 &\stackrel{?}{<} 2(0)-3 \\ 0 &\not< -3 \end{aligned}$$

Hence, $(0,0)$ doesn't make the inequality true.

Step 4. If the ordered pair is

- a solution to the inequality, i.e., makes the inequality true, then shade the side that includes the ordered pair.
- not a solution, then shade the opposite side of the boundary line.

Since the ordered pair $(0,0)$ is not a solution to the inequality, then we shade on the opposite side of the boundary line from the location of the ordered pair.



Note

Another way of graphing linear inequalities in two variables is to complete **Step 1.** and **Step 2.**, but instead of taking a test point in **Step 3.**, we can observe the inequality symbols. If the inequality has $<$ or $\leq$, then we easily shade below the boundary line, i.e., below the y-intercept. Similarly, if the inequality has $>$ or $\geq$, then we easily shade above the boundary line, i.e., above the y-intercept.

Example 3.20

Graph the inequality: $3x+2y \geq -6$.

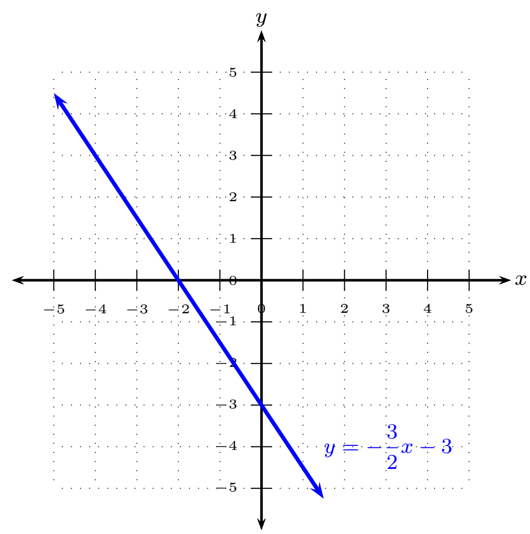
Solution.

Let's follow the steps given above to graph the inequality, but try skipping **Step 3.** and **Step 4.**

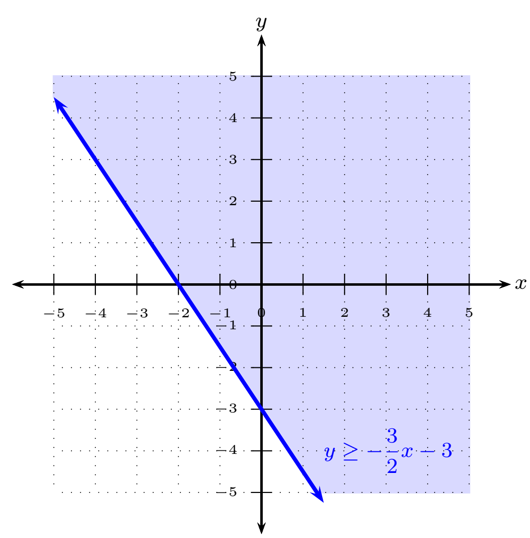
Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

$$\begin{aligned} 3x + 2y &\geq -6 \\ 2y &\geq -3x - 6 \\ y &\geq -\frac{3}{2}x - 3 \end{aligned}$$

Step 2. Graph the boundary line according to the two cases. Since the given inequality is $\geq$, then we have Case 2.



Since this inequality is $\geq$, where all ordered pairs above the boundary line are solutions to the inequality, we can easily shade above the y-intercept:



3.4.4 Linear Inequalities in Two Variables Homework

Determine whether the given ordered pairs are solutions to the inequality.

1. $x + 2y \geq -4$; $(0, -4)$; $(1, 1)$

2. $2x - y \leq 2$; $(1, 5)$; $(3, 1)$

Graph the following inequalities.

3. $2x - y \leq 2$

4. $x > 4y - 8$

5. $x + 2y \geq -4$

6. $3x + 4y < 12$

7. $6x + 8y \leq 24$

8. $5x + 3y \leq 15$

9. $y > 3x + 1$

10. $3x + 2y \leq 12$

11. $5x - 2y < 10$

12. $3x + 4y \geq 24$

13. $y \leq 3x - 4$

3.5 Lin. Inequal. in One and Two Var.: Answers to the Homework Exercises

Linear Inequalities

1. $(-5, \infty)$

3. $(-\infty, 5]$

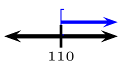
5. $(-\infty, 1]$

7. $x < 2$; $(-\infty, 2)$

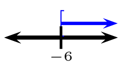
9. $x \geq 5$; $[5, \infty)$

11. $x > -2$; $(-2, \infty)$

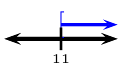
13. $[110, \infty)$



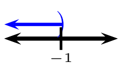
15. $[-6, \infty)$



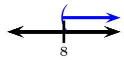
17. $[11, \infty)$



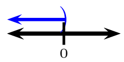
19. $(-\infty, -1)$



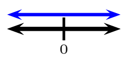
21. $(8, \infty)$



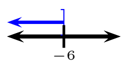
23. $(-\infty, 0)$



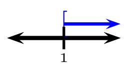
25. $(-\infty, \infty)$



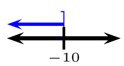
27. $(-\infty, -6]$



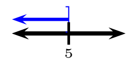
29. $[1, \infty)$



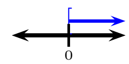
31. $(-\infty, -10]$



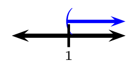
33. $(-\infty, 5]$



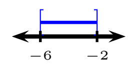
35. $[0, \infty)$



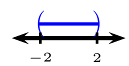
37. $(1, \infty)$



39. $[-6, -2]$

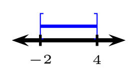


41. $(-2, 2)$



43. no solution

45. $[-2, 4]$



Compound Inequalities

1. $n \leq -9$ or $n \geq 2 : (-\infty, -9] \cup [2, \infty)$

3. $x < -7 : (-\infty, -7)$

5. $b < 5 : (-\infty, 5)$

7. $-6 \leq x \leq -2 : [-6, -2]$

9. $-2 < x < 2 : (-2, 2)$

11. No solution; $\varnothing$

13. $5 \leq x < 19 : [5, 19)$

15. $k \geq 2$ or $k < -20$: $(-\infty, -20) \cup [2, \infty)$

17. $m \geq -4$ or $m < -5$: $(-\infty, -5) \cup [-4, \infty)$

19. $n < -7$ or $n > 8$: $(-\infty - 7), \cup (8, \infty)$

21. $-2 \leq n \leq 6 : [-2, 6]$

23. $-9 \leq x \leq 0 : [-9, 0]$

25. No solution; $\varnothing$

27. $x \leq 0$ or $x > 8$: $(-\infty, 0] \cup (8, \infty)$

29. $n < -14$ or $n \geq 17$: $(-\infty, -14) \cup [17, \infty)$

31. all real numbers

Absolute Value Inequalities

1. $-3, 3$

3. $-4, 8$

5. $-\frac{7}{3}, \frac{11}{3}$

7. $1, 4$

9. $(-\infty, -6) \cup (0, \infty)$

11. $(-\infty, -1] \cup [3, \infty)$

13. $[1, 3]$

15. $\left[1, \frac{3}{2}\right]$

17. $[2, 4]$

19. $-8, 8$

21. $-4, 20$

23. $-7, 2$

25. $(-\infty, 5) \cup (5, \infty)$

27. $(-\infty, -1) \cup (5, \infty)$

29. $\left[-\frac{4}{3}, 2\right]$

31. $\left[\frac{1}{2}, 1\right]$

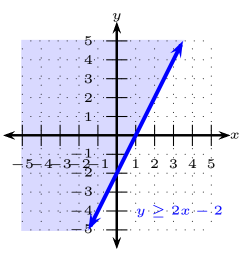
33. $\left[-2, -\frac{4}{3}\right]$

35. $[-3, -2]$

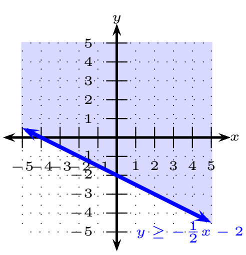
Linear Inequalities in Two Variables

1. yes; no

3.



5.



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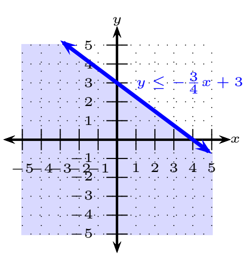

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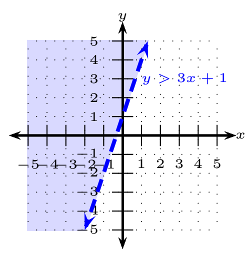

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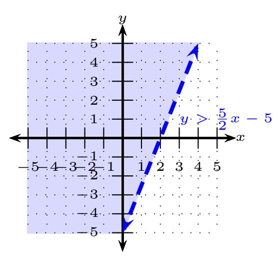

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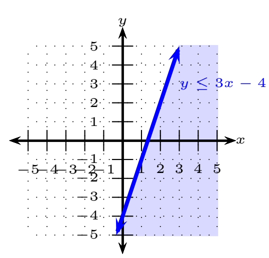


Chapter 4

Systems of Linear Equations in Two and Three Variables

Chapter Objectives

By the end of this chapter, the student should be able to

- Solve a system of equations with two and three linear equations in two and three variables by graphing, substitution, and elimination including infinitely many solutions or no solution
- Solve applications involving systems of equations including mixture, value, distance, and interest problems
- Graph and find the solutions for systems of two linear inequalities in two variables
- Use matrices to solve systems of two linear equations in two variables

We have solved linear equations like $3x - 4 = 11$ by adding 4 to both sides and then dividing by 3 (solution is $x = 5$). Notice, we only have one variable in this equation. What if we have 2 variables? Luckily, we have methods to solve equations with more than one variable. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables, such as x and y, we will need two equations with the same variables. When solving for more than one equation and one variable, we call the set of equations a *system of equations*. When solving a *system of equations*, we are looking for a solution that makes both equations true. Since we are solving for x and y, it should remind us of graphing lines, and the solution is an ordered pair (x, y). This ordered-pair is on both lines.

Definition

A ***system of two linear equations in two variables*** is given in the form

$$\begin{cases} ax + by &= c \\ dx + ey &= f \end{cases}$$

where $a, b, c, d, e,$ and f are coefficients and x and y are variables. This system is represented in *standard form.*

4.1 System of equations: graphing

Three types of solutions to a linear system by graphing

Given two linear equations and after graphing the lines,

Solution 1. If the two lines intersect, then the point of intersection is the solution to the system, i.e., the solution is an ordered-pair (x, y).

Solution 2. If the two lines do not intersect and are parallel, i.e., they have the same slope and different y-intercepts, then the system has *no solution*.

Solution 3. If the two lines are the same line, then the solution is *infinitely many solutions* on that line.

4.1.1 Verifying solutions

Example 4.1

Is the ordered-pair $(2, 1)$ a solution to the system

$$\begin{cases} 3x - y = 5 \\ x + y = 3 \end{cases} \quad ?$$

Solution.

To verify whether $(2, 1)$ is the solution to the system, we plug-n-chug $(2, 1)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for both equations in the system, then $(2, 1)$ will be the solution to the system.

$$\begin{aligned} 3x - y &= 5 && \text{Plug-n-chug } x = 2 \text{ and } y = 1 \\ 3(2) - (1) &\stackrel{?}{=} 5 && \text{Simplify} \\ 6 - 1 &\stackrel{?}{=} 5 && \text{Subtract} \\ 5 &= 5 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} x + y &= 3 && \text{Plug-n-chug } x = 2 \text{ and } y = 1 \\ (2) + (1) &\stackrel{?}{=} 3 && \text{Add} \\ 3 &= 3 && \checkmark \text{ True} \end{aligned}$$

Since the ordered-pair $(2, 1)$ makes both statements true, then $(2, 1)$ is a solution to the system. Hence, if we were to graph these lines, they would intersect at the point $(2, 1)$.

Example 4.2

Is the ordered-pair $(-3, -4)$ a solution to the system

$$\begin{cases} 5x + 4y = -31 \\ 3x + 6y = -36 \end{cases} \quad ?$$

Solution.

To verify whether $(-3,-4)$ is the solution to the system, we plug-n-chug $(-3,-4)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for both equations in the system, then $(-3,-4)$ will be the solution to the system.

$$\begin{aligned} 5x+4y &= -31 && \text{Plug-n-chug } x=-3 \text{ and } y=-4\\ 5(-3)+4(-4) &\stackrel{?}{=} -31 && \text{Simplify}\\ -15-16 &\stackrel{?}{=} -31 && \text{Subtract}\\ -31 &= -31 && \checkmark\ \text{True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} 3x+6y &= -36 && \text{Plug-n-chug } x=-3 \text{ and } y=-4\\ 3(-3)+6(-4) &\stackrel{?}{=} -36 && \text{Simplify}\\ -9-24 &\stackrel{?}{=} -36 && \text{Subtract}\\ -33 &\neq -36 && \text{✗ False} \end{aligned}$$

Since the ordered-pair $(-3,-4)$ makes only one of the statements true, then $(-3,-4)$ is not a solution to the system. Recall, the ordered-pair must make the statement true for both equations in order to be a solution to the system.

4.1.2 Solve a system by graphing

Example 4.3

Solve the system by graphing:

$$\begin{cases} y = -\dfrac{1}{2}x+3 \\[2ex] y = \dfrac{3}{4}x-2 \end{cases}$$

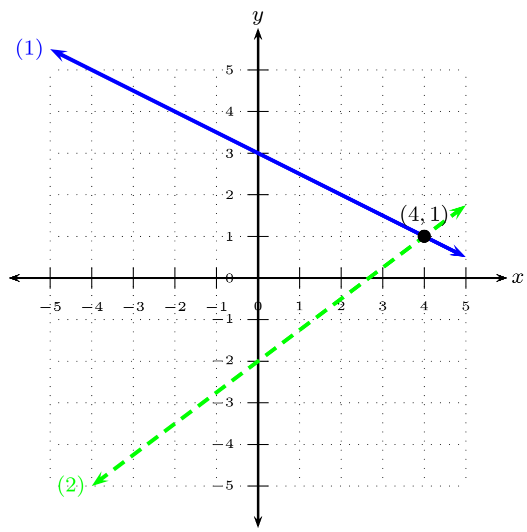
Solution.

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Notice both equations are given in slope-intercept form. Let's go ahead and use the slope-intercept form to graph the lines.

$$y = -\frac{1}{2}x+3 \qquad (1)$$

$$y = \frac{3}{4}x-2 \qquad (2)$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.



We can see after graphing the two lines that they intersect at the point $(4, 1)$. Hence, the solution to the system is $(4, 1)$.

Example 4.4

Solve the system by graphing:

$$\begin{cases} 6x - 3y = -9 \\ 2x + 2y = -6 \end{cases}$$

Solution.

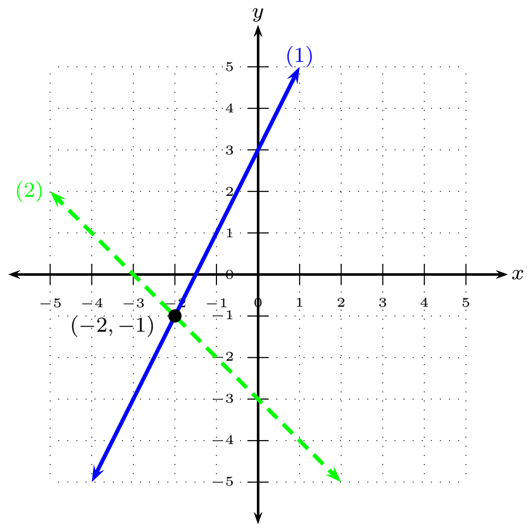
We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Since both equations are not given in slope-intercept form as in example 4.3, we can rewrite them in slope-intercept form, then graph. So, let's rewrite each equation in slope-intercept form:

$$\begin{array}{ll} 6x - 3y = -9 & 2x + 2y = -6 \\ -3y = -6x - 9 & 2y = -2x - 6 \\ y = \dfrac{-6x}{-3} - \dfrac{9}{-3} & y = \dfrac{-2}{2}x - \dfrac{6}{2} \\ & \\ y = 2x + 3 & y = -x - 3 \end{array}$$

Let's go ahead and use the slope-intercept form to graph the lines.

$$\begin{aligned} y &= 2x + 3 \qquad (1) \\ y &= -x - 3 \qquad (2) \end{aligned}$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.



We can see after graphing the two lines that they intersect at the point $(-2, -1)$. Hence, the solution to the system is $(-2, -1)$.

Example 4.5

Solve the system by graphing:

$$\begin{cases} y = \dfrac{3}{2}x - 4 \\ \\ y = \dfrac{3}{2}x + 1 \end{cases}$$

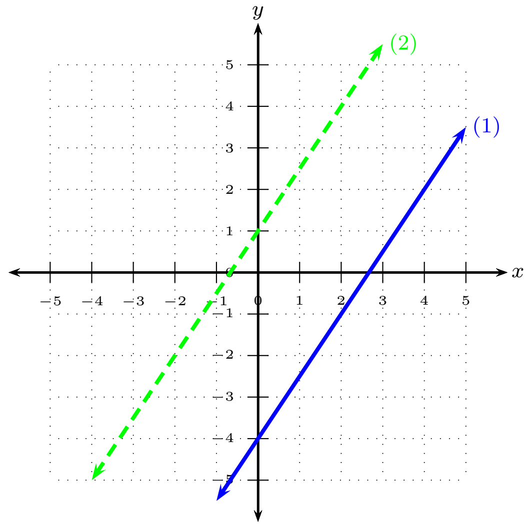
Solution.

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Notice both equations are given in slope-intercept form. Let's go ahead and use the slope-intercept form to graph the lines.

$$y = \frac{3}{2}x - 4 \qquad (1)$$

$$y = \frac{3}{2}x + 1 \qquad (2)$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.



We can see after graphing the two lines that these two lines are parallel. Hence, there is no solution to the system (since they will never intersect). Note, we could see by the system that these lines shared the same slope, but had different y-intercepts. Without graphing, we could have seen that these lines were parallel, hence, having no solution.

Example 4.6

Solve the system by graphing:

$$\begin{cases} 2x - 6y = 12 \\ 3x - 9y = 18 \end{cases}$$

Solution.

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Since neither of the equations are written in slope-intercept form, let's try graphing by making a table for each equation. Start with equation (1):

$$x = -3 \qquad \begin{aligned} 2(-3) - 6y &= 12 \\ -6 - 6y &= 12 \\ -6y &= 18 \\ y &= -3 \end{aligned}$$

$$x = 0 \qquad \begin{aligned} 2(0) - 6y &= 12 \\ -6y &= 12 \\ y &= -2 \end{aligned}$$

$$x = 3 \qquad \begin{aligned} 2(3) - 6y &= 12 \\ 6 - 6y &= 12 \\ -6y &= 6 \\ y &= -1 \end{aligned}$$

x	y
-3	-3
0	-2
3	-1

Next, equation (2):

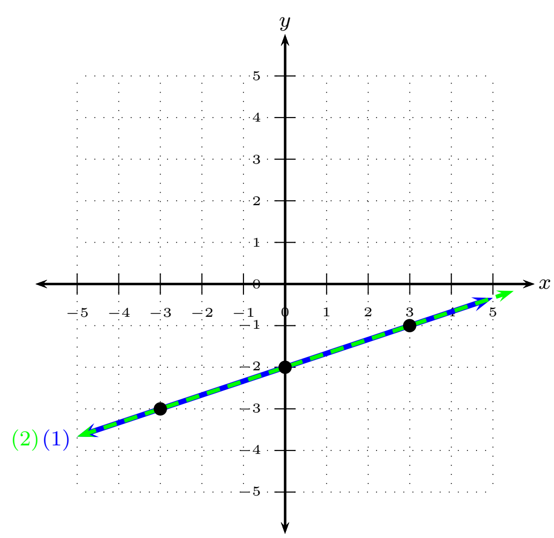
$$x = -3 \qquad \begin{aligned} 3(-3) - 9y &= 18 \\ -9 - 9y &= 18 \\ -9y &= 27 \\ y &= -3 \end{aligned}$$

x	y
-3	-3
0	-2
3	-1

$$x = 0 \qquad \begin{aligned} 3(0) - 9y &= 18 \\ -9y &= 18 \\ y &= -2 \end{aligned}$$

$$x = 3 \qquad \begin{aligned} 3(3) - 9y &= 18 \\ 9 - 9y &= 18 \\ -9y &= 9 \\ y &= -1 \end{aligned}$$

Now, let's graph the ordered-pairs. We will graph line (1) with a solid line and graph line (2) with a dashed line.



We can see after graphing the two lines that these two lines are the same. Hence, there are infinitely many solutions on the line $2x - 6y = 12$ (or the other equation) to the system (since they intersect at every point on the line). Note, we could see by the system that these lines shared the same slope and y-intercepts (after putting them in slope-intercept form). Without graphing, we could have seen that these lines were the same line, hence, having infinitely many solutions.

World Note

The Babylonians were the first to work with systems of equations with two variables. However, their work with systems was quickly passed by the Greeks, around 300 AD, who would solve systems of equations with three or four variables and eventually developed methods for solving systems with any number of unknowns.

4.1.3 System of Equations: Graphing Homework

Determine whether the given ordered pair(s) is a solution to the system.

1. $2x + 8y = 0$
$-8x + 3y = 38$; $(-4, 1)$

2. $-5x + 6y = 11$
$4x + 2y = -4$; $(-1, 1)$

3. $6x + 5y = 49$
$-x - 6y = -34$; $(4, 5)$

4. $-2x + 2y = -8$
$-6x - 3y = -6$; $(2, -2)$

Solve each system by graphing.

1. $y = -x + 1$
$y = -5x - 3$

2. $y = -3$
$y = -x - 4$

3. $y = -\frac{3}{4}x + 1$
$y = -\frac{3}{4}x + 2$

4. $y = \frac{1}{3}x + 2$
$y = -\frac{5}{3}x - 4$

5. $y = \frac{5}{3}x + 4$
$y = -\frac{2}{3}x - 3$

6. $x + 3y = -9$
$5x + 3y = 3$

7. $x - y = 4$
$2x + y = -1$

8. $2x + 3y = -6$
$2x + y = 2$

9. $2x + y = 2$
$x - y = 4$

10. $2x + y = -2$
$x + 3y = 9$

11. $0 = -6x - 9y + 36$
$12 = 6x - 3y$

12. $2x - y = -1$
$0 = -2x - y - 3$

13. $3 + y = -x$
$-4 - 6x = -y$

14. $-y + 7x = 4$
$-y - 3 + 7x = 0$

15. $-12 + x = 4y$
$12 - 5x = 4y$

16. $y = -\frac{5}{4}x - 2$
$y = -\frac{1}{4}x + 2$

17. $y = -x - 2$
$y = \frac{2}{3}x + 3$

18. $y = 2x + 2$
$y = -x - 4$

19. $y = 2x - 4$
$y = \frac{1}{2}x + 2$

20. $y = \frac{1}{2}x + 4$
$y = \frac{1}{2}x + 1$

21. $x + 4y = -12$
$2x + y = 4$

22. $6x + y = -3$
$x + y = 2$

23. $3x + 2y = 2$
$3x + 2y = -6$

24. $x + 2y = 6$
$5x - 4y = 16$

25. $x - y = 3$
$5x + 2y = 8$

26. $-2y + x = 4$
$2 = -x + \frac{1}{2}y$

27. $-2y = -4 - x$
$-2y = -5x + 4$

28. $16 = -x - 4y$
$-2x = -4 - 4y$

29. $-4 + y = x$
$x + 2 = -y$

30. $-5x + 1 = -y$
$-y + x = -3$

4.2 Systems of equations: the substitution method

Solving a system by graphing has its limitations. First, it requires the graph to be precisely drawn. If the lines aren't drawn correctly, it is possible to obtain the incorrect point of intersection. Second, graphing is not a sufficient method to use if the point of intersection is far away from the origin, like the ordered-pair $(12567, 9832)$, or if the answer is a decimal (or irrational number) that the graph will not display well. For these reasons, we rarely use graphing to solve systems. Instead, we use an algebraic approach. There are two approaches and the first approach is called substitution. We build the concepts of substitution through several examples and then conclude with a general four-step process to solve problems using this method.

4.2.1 Introduction to substitution

Example 4.7

Solve the system of equations.

$$\begin{cases} x = 5 \\ y = 2x - 3 \end{cases}$$

Solution.

We are given $x = 5$ in the first equation. Hence, we can plug-n-chug $x = 5$ into the second equation to find y because we know any ordered-pair with x coordinate 5 will satisfy the equation:

$$\begin{aligned} y &= 2x - 3 && \text{Plug-n-chug } x = 5 \\ y &= 2(5) - 3 && \text{Evaluate} \\ y &= 10 - 3 && \text{Subtract} \\ y &= 7 && y\text{-coordinate of the solution} \end{aligned}$$

Since it is given $x = 5$ and we obtained $y = 7$, then the solution to the system is the ordered-pair $(5, 7)$. Furthermore, if we were to graph these two lines, they would intersect at $(5, 7)$.

4.2.2 The substitution method

When we are given the solution to one of the two variables, we can easily plug-n-chug that value (or expression) in the other equation to obtain the value of the second variable. It is very important that when we substitute, we write parenthesis around the expression we are substituting; this will minimize errors with distribution.

Example 4.8

Solve the system by substitution.

$$\begin{cases} 2x - 3y = 7 \\ y = 3x - 7 \end{cases}$$

Solution.

We can see that y is isolated in the second equation, $y = 3x - 7$, and we can substitute the right side for y in the first equation.

$$\begin{aligned} 2x-3y&=7 && \text{Plug-n-chug } y=3x-7 \text{ into the first equation}\\ 2x-3(3x-7)&=7 && \text{Distribute}\\ 2x-9x+21&=7 && \text{Combine like terms}\\ -7x+21&=7 && \text{Isolate the variable term}\\ -7x&=-14 && \text{Multiply by the reciprocal of } -7\\ x&=2 && x\text{-coordinate of the solution} \end{aligned}$$

Since $x = 2$, then we can plug-n-chug $x = 2$ into one of the equations to obtain y:

$$\begin{aligned} y&=3x-7 && \text{Plug-n-chug } x=2\\ y&=3(2)-7 && \text{Evaluate}\\ y&=6-7 && \text{Subtract}\\ y&=-1 && y\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(2, -1)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(2, -1)$. When we obtain a solution to a system, we call this system a *consistent system*. When we obtain one ordered-pair as the solution to the system, we call this solution an *independent solution*.

Definition

Given a system of equations,

- If there exists at least one solution to the system, then this is a ***consistent system***.
  - If the consistent system has one unique solution, i.e., the solution is an ordered-pair, then this is an ***independent system***.
  - If the consistent system has more than one solution, i.e., there are infinitely many solutions, then this is a ***dependent system***.
- If a solution to the system doesn't exist, i.e., there is no solution, then this is an ***inconsistent system***.

4.2.3 Solve for a variable

Example 4.9

Solve the system by substitution:

$$\begin{cases} 3x+2y=1\\ x-5y=6 \end{cases}$$

Solution.

Notice neither of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation. We always want to work smarter, not harder, so let's be clever in choosing the equation and variable. Looking at the first equation, there is a coefficient in front of each variable. Looking at the second equation, we see only the y has a coefficient other than 1, and x's coefficient is one. Let's pick this because solving for x

is a one-step equation while the other is a two-step equation.

$$\begin{aligned} x - 5y &= 6 \\ x &= 6 + 5y \end{aligned}$$

Now, we can substitute x into the first equation:

$$\begin{aligned} 3x + 2y &= 1 && \text{Plug-n-chug } x = 6 + 5y \\ 3(6 + 5y) + 2y &= 1 && \text{Distribute} \\ 18 + 15y + 2y &= 1 && \text{Combine like terms} \\ 18 + 17y &= 1 && \text{Isolate the variable term} \\ 17y &= -17 && \text{Multiply by the reciprocal of 17} \\ y &= -1 && y\text{-coordinate of the solution} \end{aligned}$$

Since $y = -1$, then we can plug-n-chug $y = -1$ into one of the equations to obtain x:

$$\begin{aligned} x &= 6 + 5y && \text{Plug-n-chug } y = -1 \\ x &= 6 + 5(-1) && \text{Evaluate} \\ x &= 6 - 5 && \text{Subtract} \\ x &= 1 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(1, -1)$. Furthermore, if we were to graph these two lines, they would intersect at $(1, -1)$. Also, we know this system is a consistent system that is independent.

✎ Note

When choosing the equation and variable to use in substitution, *we want to choose the one-step equation*, i.e., one equation with one of the variables with a coefficient of one. Even though we are used to always solving for y when graphing lines, with this method, it doesn't matter. We want to work smarter, not harder, and pick the equation that will give us the solution the quickest.

The substitution method

Given a system of two linear equations in two variables, we can use the following steps to solve by substitution.

Step 1. Choose an equation and then solve for x or y. (*Choose the one-step equation when possible.*)

Step 2. Substitute the expression for x or y in the other equation.

Step 3. Solve the equation.

Step 4. Substitute the value into one of the original equations to find the remaining variable.

It is common practice to write your answer as an ordered-pair of the form (x, y) since this is the point of intersection. Be sure to verify the solution.

Example 4.10

Solve the system by substitution:

$$\begin{cases} 4x - 2y = 2 \\ 2x + y = -5 \end{cases}$$

Solution.

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the second equation, we see only the y has a coefficient of one, and x's coefficient is two. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} 2x + y &= -5 \\ y &= -2x - 5 \end{aligned}$$

Step 2. Now, we can substitute y into the first equation:

$$\begin{aligned} 4x - 2y &= 2 && \text{Plug-n-chug } y = -2x - 5 \\ 4x - 2(-2x - 5) &= 2 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 4x - 2(-2x - 5) &= 2 && \text{Distribute} \\ 4x + 4x + 10 &= 2 && \text{Combine like terms} \\ 8x + 10 &= 2 && \text{Isolate the variable term} \\ 8x &= -8 && \text{Multiply by the reciprocal of 8} \\ x &= -1 && x\text{-coordinate of the solution} \end{aligned}$$

Step 4. Since $x = -1$, then we can plug-n-chug $x = -1$ into one of the equations to obtain y:

$$\begin{aligned} y &= -2x - 5 && \text{Plug-n-chug } x = -1 \\ y &= -2(-1) - 5 && \text{Evaluate} \\ y &= 2 - 5 && \text{Subtract} \\ y &= -3 && y\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(-1, -3)$. Furthermore, if we were to graph these two lines, they would intersect at $(-1, -3)$. Also, we know this system is a consistent system that is independent.

4.2.4 Substitution: special cases

Example 4.11

Solve the system by substitution:

$$\begin{cases} y + 4 = 3x \\ 2y - 6x = -8 \end{cases}$$

Solution.

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the first equation, we see only the y has a coefficient of one, and x's coefficient is three. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} y+4&=3x \\ y&=3x-4 \end{aligned}$$

Step 2. Now, we can substitute y into the second equation:

$$\begin{aligned} 2y-6x&=-8 && \text{Plug-n-chug } y=3x-4 \\ 2(3x-4)-6x&=-8 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 2(3x-4)-6x&=-8 && \text{Distribute} \\ 6x-8-6x&=-8 && \text{Combine like terms} \\ -8&=-8 \end{aligned}$$

Since all the variables cancel and we are left with a statement without variables, we ask," Is this statement true?"

$$\begin{aligned} -8&\stackrel{?}{=}-8 && \text{Is this true?} \\ -8&=-8 && \checkmark\ \text{True} \end{aligned}$$

Since this statement is true, then there are infinitely many solutions on the line $y+4=3x$. Furthermore, if we were to graph these two lines, we know they would be the same line and intersect at every point on the line. Also, we know this system is a consistent system that is dependent.

Example 4.12

Solve the system by substitution:

$$\begin{cases} 6x-3y=-9 \\ -2x+y=5 \end{cases}$$

Solution.

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the second equation, we see only the y has a coefficient of one, and x's coefficient is -2. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} -2x+y&=5 \\ y&=2x+5 \end{aligned}$$

Step 2. Now, we can substitute y into the first equation:

$$\begin{aligned} 6x - 3y &= -9 \qquad \text{Plug-n-chug } y = 2x+5 \\ 6x - 3(2x+5) &= -9 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 6x - 3(2x+5) &= -9 \qquad \text{Distribute} \\ 6x - 6x - 15 &= -9 \qquad \text{Combine like terms} \\ -15 &= -9 \end{aligned}$$

Since all the variables cancel and we are left with a statement without variables, we ask,“ Is this statement true?”

$$\begin{aligned} -15 &\stackrel{?}{=} -9 \qquad \text{Is this true?} \\ -15 &\neq -9 \qquad \text{✗False} \end{aligned}$$

Since this statement is false, then there is no solution to this system. Furthermore, if we were to graph these two lines, we know they would be parallel. Hence, this system is an inconsistent system.

World Note

French mathematician *Rene Descartes* wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is the reason we are often solving for the variables x, y, and z.

Example 4.13

Solve the system by substitution:

$$\begin{cases} 5x - 6y = -14 \\ -2x + 4y = 12 \end{cases}$$

Solution.

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at both equations, we see none of these are one-step equations. Hence, we can just pick one equation and solve for a variable. Notice in the second equation, all coefficients are divisible by 2. Let's pick this because solving for x would avoid fractions.

$$\begin{aligned} -2x + 4y &= 12 \\ -2x &= -4y + 12 \\ x &= 2y - 6 \end{aligned}$$

Step 2. Now, we can substitute x into the first equation:

$$\begin{aligned} 5x - 6y &= -14 \qquad \text{Plug-n-chug } x = 2y-6 \\ 5(2y-6) - 6y &= -14 \end{aligned}$$

Step 3. Solve for y:

$$\begin{aligned} 5(2y-6)-6y &= -14 && \text{Distribute} \\ 10y-30-6y &= -14 && \text{Combine like terms} \\ 4y-30 &= -14 && \text{Isolate the variable term} \\ 4y &= 16 && \text{Multiply by the reciprocal of 4} \\ y &= 4 && y\text{-coordinate of the solution} \end{aligned}$$

Step 4. Since $y=4$, then we can plug-n-chug $y=4$ into one of the equations to obtain x:

$$\begin{aligned} x &= 2y-6 && \text{Plug-n-chug } y=4 \\ x &= 2(4)-6 && \text{Evaluate} \\ x &= 8-6 && \text{Subtract} \\ x &= 2 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(2,4)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(2,4)$. Also, we know this system is a consistent system that is independent .

4.2.5 Systems of Equations: The Substitution Method Homework

Solve each system by substitution. Determine if each system is consistent, independent or dependent, or inconsistent.

1. $y = -3x$, $y = 6x - 9$

2. $y = -2x - 9$, $y = 2x - 1$

3. $y = 6x + 4$, $y = -3x - 5$

4. $y = 3x + 2$, $y = -3x + 8$

5. $y = 2x - 3$, $y = -2x + 9$

6. $y = 6x - 6$, $-3x - 3y = -24$

7. $y = -6$, $3x - 6y = 30$

8. $y = -5$, $3x + 4y = -17$

9. $-2x + 2y = 18$, $y = 7x + 15$

10. $y = -8x + 19$, $-x + 6y = 16$

11. $7x - 2y = -7$, $y = 7$

12. $x - 5y = 7$, $2x + 7y = -20$

13. $-2x - y = -5$, $x - 8y = -23$

14. $-6x + y = 20$, $-3x - 3y = -18$

15. $3x + y = 9$, $2x + 8y = -16$

16. $y = x + 5$, $y = -2x - 4$

17. $y = -6x + 3$, $y = 6x + 3$

18. $y = 3x + 13$, $y = -2x - 22$

19. $y = -2x - 9$, $y = -5x - 21$

20. $y = 7x - 24$, $y = -3x + 16$

21. $-x + 3y = 12$, $y = 6x + 21$

22. $6x - 4y = -8$, $y = -6x + 2$

23. $7x + 2y = -7$, $y = 5x + 5$

24. $y = x + 4$, $3x - 4y = -19$

25. $y = -2x + 8$, $-7x - 6y = -8$

26. $x - 2y = -13$, $4x + 2y = 18$

27. $3x - 4y = 15$, $7x + y = 4$

28. $6x + 4y = 16$, $-2x + y = -3$

29. $7x + 5y = -13$, $x - 4y = -16$

30. $-5x - 5y = -20$, $-2x + y = 7$

31. $2x + y = 2$, $3x + 7y = 14$

32. $x + 5y = 15$, $-3x + 2y = 6$

33. $-2x + 4y = -16$, $y = -2$

34. $-6x + 6y = -12$, $8x - 3y = 16$

35. $2x + 3y = 16$, $-7x - y = 20$

36. $2x + y = -7$, $5x + 3y = -21$

37. $2x + 3y = -10$, $7x + y = 3$

38. $-2x + 2y = -22$, $-5x - 7y = -19$

39. $-8x + 2y = -6$, $-2x + 3y = 11$

40. $-x - 4y = -14$, $-6x + 8y = 12$

4.3 System of equations: the addition method

We first found that graphing isn't a sophisticated way for generally solving systems. We then considered a second method known as substitution. The substitution method is often used for solving systems in various areas of algebra. However, substitution can get quite involved, especially if there are fractions because this only allows more room for error. Hence, we need an even more sophisticated way for solving systems in general. We call this method the *addition method*, also called the elimination method. We will build the concept in the following examples, then define a four-step process we can use to solve by elimination.

4.3.1 The addition method

Example 4.14

Solve the system by addition (elimination).

$$\begin{cases} 3x - 4y = 8 \\ 5x + 4y = -24 \end{cases}$$

Solution.

We solve the system by addition because we do just that- add. We want to add the two equations together to obtain an equation of one variable. Hence, we cannot just add right away; we need to make sure that when we add, we will eliminate one of the variables. Looking at the y variable terms, we can see that the coefficients of y are the same but opposite signs. We can foresee that when we add these two equations, the y variable terms will cancel:

$$\begin{array}{rr} & 3x - 4y = 8 \\ + & 5x + 4y = -24 \\ \hline & 8x = -16 \end{array} \qquad \text{Add the equations}$$

Notice the y variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{aligned} 8x &= -16 && \text{Multiply by the reciprocal of } 8 \\ x &= -2 && x\text{-coordinate of the solution} \end{aligned}$$

Since $x = -2$, then we can plug-n-chug $x = -2$ into one of the equations to obtain y:

$$\begin{aligned} 3x - 4y &= 8 && \text{Plug-n-chug } x = -2 \\ 3(-2) - 4y &= 8 && \text{Evaluate} \\ -6 - 4y &= 8 && \text{Isolate the variable term} \\ -4y &= 14 && \text{Multiply by the reciprocal of } -4 \\ y &= -\frac{14}{4} && \text{Reduce the fraction} \\ y &= -\frac{7}{2} && y\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $\left(-2, -\frac{7}{2}\right)$. Furthermore, if we were to graph these two lines, we know they would intersect at $\left(-2, -\frac{7}{2}\right)$. Also, we know this system is a consistent system with an independent solution.

4.3.2 The addition method with multiplication

In example 4.14, one of the variable terms had the same coefficient, but opposite signs, and adding these together eliminated the variable terms completely, which allowed us to solve for the other variable. This is the idea behind the addition method. However, generally, we aren't given variable terms that have the same coefficient with opposite signs. So, we will manipulate the equations by multiplying one or both equations by the LCM of the coefficients for one of the variables. We want to work smarter, not harder, so we should be clever in which variable we choose.

Example 4.15

Solve the system by addition (elimination).

$$\begin{cases} -6x + 5y = 22 \\ 2x + 3y = 2 \end{cases}$$

Solution.

Since none of the variable terms have the same coefficient with opposite signs, we need to choose a variable and rewrite the equations so we can cancel the variable. Recall, the goal is to obtain an equation in one variable after adding. Looking at the x variable terms, we can see that the coefficients of x have opposite signs. So let's choose to eliminate x and we multiply the second equation by 3 to obtain the $\text{LCM}(2,6) = 6$:

$$\begin{aligned} 3 \cdot (2x + 3y) &= (2) \cdot 3 && \text{Distribute} \\ 6x + 9y &= 6 \end{aligned}$$

Notice the x variable terms have the same coefficients with opposite signs. Now we can add and eliminate x:

$$\begin{array}{rr} & -6x + 5y = 22 \\ + & 6x + 9y = 6 \\ \hline & 14y = 28 \end{array} \qquad \text{Add the equations}$$

Notice the x variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{aligned} 14y &= 28 && \text{Multiply by the reciprocal of 14} \\ y &= 2 && y\text{-coordinate of the solution} \end{aligned}$$

Since $y = 2$, then we can plug-n-chug $y = 2$ into one of the equations to obtain x:

$$\begin{aligned} 2x + 3y &= 2 && \text{Plug-n-chug } y = 2 \\ 2x + 3(2) &= 2 && \text{Evaluate} \\ 2x + 6 &= 2 && \text{Isolate the variable term} \\ 2x &= -4 && \text{Multiply by the reciprocal of 2} \\ x &= -2 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(-2, 2)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(-2, 2)$. Also, we know this system is a consistent system with an independent solution.

4.3.3 Multiplying two equations

Example 4.16

Solve the system by addition (elimination).

$$\begin{cases} 3x + 6y = -9 \\ 2x + 9y = -26 \end{cases}$$

Solution.

Since none of the variable terms have the same coefficient with opposite signs, we need to choose a variable and rewrite the equations so we can cancel the variable. Recall, the goal is to obtain an equation in one variable after adding. Looking at the x and y variable terms, we can see that none of the coefficients are the same or with opposite signs. So we can choose any variable to eliminate. Let's choose to eliminate y and we multiply both equations by a factor to obtain the LCM$(9, 6) = 18$ with opposite signs:

$$\begin{aligned} -3 \cdot (3x + 6y) &= (-9) \cdot -3 \\ 2 \cdot (2x + 9y) &= (-26) \cdot 2 \end{aligned}$$

Notice the y variable terms have the same coefficients with opposite signs

$$\begin{aligned} -9x - 18y &= 27 \\ 4x + 18y &= -52 \end{aligned}$$

Now we can add and eliminate y:

$$\begin{array}{rr} & -9x - 18y = 27 \\ + & 4x + 18y = -52 \\ \hline & -5x = -25 \end{array} \qquad \text{Add the equations}$$

Notice the y variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{aligned} -5x &= -25 && \text{Multiply by the reciprocal of } -5 \\ x &= 5 && x\text{-coordinate of the solution} \end{aligned}$$

Since $x = 5$, then we can plug-n-chug $x = 5$ into one of the equations to obtain y:

$$\begin{aligned} 2x + 9y &= -26 && \text{Plug-n-chug } x = 5 \\ 2(5) + 9y &= -26 && \text{Evaluate} \\ 10 + 9y &= -26 && \text{Isolate the variable term} \\ 9y &= -36 && \text{Multiply by the reciprocal of } 9 \\ y &= -4 && y\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(5, -4)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(5, -4)$. Also, we know this system is a consistent system with an independent solution.

The addition method

Given a system of two linear equations in two variables, we can use the following steps to solve by addition (elimination).

Step 1. Choose a variable to eliminate. (*Choose the variable with the variable terms with opposite signs, same coefficient, or both, if possible.*)

Step 2. Multiply one or both equations so that the coefficients of this variable are the LCM of the coefficients with opposite signs.

Step 3. Add the equations together, then solve.

Step 4. Substitute the value into one of the original equations to find the remaining variable.

It is common practice to write your answer as an ordered-pair of the form (x, y) since this is the point of intersection. Be sure to verify the solution.

Example 4.17

Solve the system by addition (elimination):

$$\begin{cases} 2x - 5y = -13 \\ 5x - 3y = -4 \end{cases}$$

Solution.

Since none of the variable terms have the same coefficient, opposite signs, or both, we need to choose a variable and rewrite the equations so we can cancel the variable. We can choose any variable to eliminate.

Step 1. Let's choose to eliminate x and we multiply both equations by a factor to obtain the $\text{LCM}(2, 5) = 10$ with opposite signs.

Step 2. We can mulitply the first equation by a factor of 5 and the second equation by a factor of -2 so that we obtain variable terms with the same coefficient, 10, with opposite signs:

$$\begin{aligned} 5 \cdot (2x - 5y) &= (-13) \cdot 5 \\ -2 \cdot (5x - 3y) &= (-4) \cdot -2 \end{aligned}$$

Notice the x variable terms have the same coefficients with opposite signs:

$$\begin{aligned} 10x - 25y &= -65 \\ -10x + 6y &= 8 \end{aligned}$$

Step 3. Now we can add and eliminate x:

$$\begin{array}{rr} & 10x - 25y = -65 \\ + & -10x + 6y = 8 \\ \hline & -19y = -57 \end{array} \qquad \text{Add the equations}$$

Notice the x variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{aligned} -19y &= -57 && \text{Multiply by the reciprocal of } -19 \\ y &= 3 && y\text{-coordinate of the solution} \end{aligned}$$

Step 4. Since $y = 3$, then we can plug-n-chug $y = 3$ into one of the equations to obtain x:

$$\begin{aligned} 5x - 3y &= -4 && \text{Plug-n-chug } y = 3 \\ 5x - 3(3) &= -4 && \text{Evaluate} \\ 5x - 9 &= -4 && \text{Isolate the variable term} \\ 5x &= 5 && \text{Multiply by the reciprocal of 5} \\ x &= 1 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(1, 3)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(1, 3)$. Also, we know this system is a consistent system with an independent solution.

World Note

The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 AD in China, describes a process very similar to Gaussian elimination which is very similar to the addition method.

4.3.4 Addition: special cases

Just as with graphing and substitution, it is possible to have no solution or infinite solutions with elimination. Just as with substitution, if the variables eliminate, a true statement will indicate infinitely many solutions and a false statement will indicate there is no solution.

Example 4.18

Solve the system by addition (elimination):

$$\begin{cases} 2x - 5y = 3 \\ -6x + 15y = -9 \end{cases}$$

Solution.

Since none of the variable terms have the same coefficient, but both have opposite signs, we can choose any variable to eliminate.

Step 1. Let's choose to eliminate y and we multiply both equations by a factor to obtain the $\text{LCM}(5, 15) = 15$ with opposite signs.

Step 2. We can mulitply the first equation by a factor of 3 and leave the second equation alone so that we obtain variable terms with the same coefficient, 15, with opposite signs:

$$\begin{aligned} 3 \cdot (2x - 5y) &= (3) \cdot 3 && \text{Distribute} \\ 6x - 15y &= 9 \end{aligned}$$

Notice the y variable terms have the same coefficients with opposite signs

$$\begin{aligned} 6x - 5y = 9 \\ -6x + 15y = -9 \end{aligned}$$

Step 3. Now we can add and eliminate x:

$$\begin{array}{rr} & 6x - 15y = 9 \\ + & -6x + 15y = -9 \\ \hline & 0 = 0 \end{array} \qquad \text{Add the equations}$$

Since all the variables cancel and we are left with a statement without variables, we ask,“ Is this statement true?”

$$\begin{array}{ll} 0 \stackrel{?}{=} 0 & \text{Is this true?} \\ 0 = 0 & \checkmark\ \text{True} \end{array}$$

Since this statement is true, then there are infinitely many solutions to this system. Furthermore, if we were to graph these two lines, we know they would be the same line. Hence, this system is a consistent system with a dependent solution.

Example 4.19

Solve the system by addition (elimination):

$$\begin{cases} 4x - 6y = 8 \\ 6x - 9y = 15 \end{cases}$$

Solution.

Since none of the variable terms have the same coefficient, opposite signs, or both, we need to choose a variable and rewrite the equations so we can cancel the variable. We can choose any variable to eliminate.

Step 1. Let’s choose to eliminate x and we multiply both equations by a factor to obtain the LCM$(4, 6) = 12$ with opposite signs.

Step 2. We can mulitply the first equation by a factor of 3 and the second equation by -2 so that we obtain variable terms with the same coefficient, 12, with opposite signs:

$$\begin{aligned} 3 \cdot (4x - 6y) &= (8) \cdot 3 \\ -2 \cdot (6x - 9y) &= (15) \cdot -2 \end{aligned}$$

Notice the y variable terms have the same coefficients with opposite signs:

$$\begin{aligned} 12x - 18y &= 24 \\ -12x + 18y &= -20 \end{aligned}$$

Step 3. Now we can add and eliminate x:

$$\begin{array}{rr} & 12x - 18y = 24 \\ + & -12x + 18y = -20 \\ \hline & 0 = 4 \end{array} \qquad \text{Add the equations}$$

Since all the variables cancel and we are left with a statement without variables, we ask,“ Is this statement true?”

$$\begin{array}{ll} 0 \stackrel{?}{=} 4 & \text{Is this true?} \\ 0 \neq 4 & \text{✗False} \end{array}$$

> Since this statement is false, then there is no solution to this system. Furthermore, if we were to graph these two lines, we know they would be parallel. Hence, this system is an inconsistent system.

We discussed three different methods that can be used to solve a system of two equations in two variables. While all three can be used to solve any system, graphing works great for small integer solutions. Substitution works great when we have a given variable term with a coefficient of one, and addition works great for all other cases. As each method has its own strengths, it is important to be familiar with all three methods. Next, we use these methods to solve application problems.

4.3.5 System of Equations: The Addition Method Homework

Solve each system by addition (elimination). Determine if each system is consistent, independent or dependent, or inconsistent.

1. $4x + 2y = 0$
$-4x - 9y = -28$

2. $-9x + 5y = -22$
$9x - 5y = 13$

3. $-6x + 9y = 3$
$6x - 9y = -9$

4. $4x - 6y = -10$
$4x - 6y = -14$

5. $-x - 5y = 28$
$-x + 4y = -17$

6. $2x - y = 5$
$5x + 2y = -28$

7. $10x + 6y = 24$
$-6x + y = 4$

8. $2x + 4y = 24$
$4x - 12y = 8$

9. $-7x + 4y = -4$
$10x - 8y = -8$

10. $5x + 10y = 20$
$-6x - 5y = -3$

11. $-7x - 3y = 12$
$-6x - 5y = 20$

12. $9x - 2y = -18$
$5x - 7y = -10$

13. $9x + 6y = -21$
$-10x - 9y = 28$

14. $-7x + 5y = -8$
$-3x - 3y = 12$

15. $-8x - 8y = -8$
$10x + 9y = 1$

16. $9y = 7 - x$
$-18y + 4x = -26$

17. $0 = 9x + 5y$
$y = \frac{2}{7}x$

18. $-7x + y = -10$
$-9x - y = -22$

19. $-x - 2y = -7$
$x + 2y = 7$

20. $5x - 5y = -15$
$5x - 5y = -15$

21. $-3x + 3y = -12$
$-3x + 9y = -24$

22. $-10x - 5y = 0$
$-10x - 10y = -30$

23. $-5x + 6y = -17$
$x - 2y = 5$

24. $x + 3y = -1$
$10x + 6y = -10$

25. $-6x + 4y = 12$
$12x + 6y = 18$

26. $-6x + 4y = 4$
$-3x - y = 26$

27. $-9x - 5y = -19$
$3x - 7y = -11$

28. $-5x + 4y = 4$
$-7x - 10y = -10$

29. $3x + 7y = -8$
$4x + 6y = -4$

30. $-4x - 5y = 12$
$-10x + 6y = 30$

31. $8x + 7y = -24$
$6x + 3y = -18$

32. $-7x + 10y = 13$
$4x + 9y = 22$

33. $0 = -9x - 21 + 12y$
$1 + \frac{4}{3}y + \frac{7}{3}x = 0$

34. $-6 - 42y = -12x$
$x - \frac{1}{2} - \frac{7}{2}y = 0$

4.4 Applications with systems of equations

We saw these types of examples in a previous chapter, but with one variable. In this section, we review the same types of applications, but solving in a more sophisticated way using systems of equations. Once we set up the system, we can solve using any method we choose. However, setting up the system may be the challenge, but as long as we follow the method we used before, we will be fine. We use tables to organize the parameters.

4.4.1 Value & interest problems

Example 4.20

There were 41 tickets sold for an event. Tickets for children cost \$1.50 and tickets for adults cost \$2.00. Total receipts for the event were \$73.50. How many of each type of ticket were sold?

Solution.

First, we can make a table to organize the given information and then create an equation. Let c represent the number of children tickets sold and a represent the number of adult tickets sold.

	Amount	**Value (in \$)**	**Total value**
Adult tickets	a	\$2	$2a$
Children tickets	c	\$1.50	$1.5c$
Total	41		\$73.50

Now let's set up the system. The total number of tickets sold was 41 and the total revenue from the tickets was \$73.50, and so we obtain the system

$$\begin{aligned} a + c &= 41 \\ 2a + 1.5c &= 73.50 \end{aligned}$$

At this point, we can solve using any method we choose. Since the coefficient of a and c in the first equation are both one, then let's use the method of substitution. We will solve for a in the first equation:

$$\begin{aligned} a + c &= 41 \\ a &= 41 - c \end{aligned}$$

Now, we can substitute a into the second equation and solve:

$$\begin{aligned} 2a + 1.5c &= 73.50 && \text{Plug-n-chug } a = 41 - c \\ 2(41 - c) + 1.5c &= 73.50 && \text{Distribute} \\ 82 - 2c + 1.5c &= 73.50 && \text{Combine like terms} \\ 82 - 0.5c &= 73.50 && \text{Isolate the variable term} \\ -0.5c &= -8.50 && \text{Multiply by the reciprocal of } -0.5 \\ c &= 17 && \text{Number of children tickets} \end{aligned}$$

Since $c = 17$, then we can plug-n-chug $c = 17$ into one of the equations to obtain a:

$$\begin{aligned} a &= 41 - c && \text{Plug-n-chug } c = 17 \\ a &= 41 - (17) && \text{Evaluate} \\ a &= 24 && \text{Number of adult tickets} \end{aligned}$$

Thus, there were 17 children tickets and 24 adult tickets sold.

Example 4.21

Aaron invests \$9,700 in two different accounts. The first account paid 7%, the second account paid 11% in interest. At the end of the first year he had earned \$863 in interest. How much was in each account?

Solution.

First, we can make a table to organize the given information and then create an equation. Let x represent the amount of investment in the first account and y represent the amount of investment in the second account.

	Principal	**rate**	**Total interest**
Account 1	x	0.07	$0.07x$
Account 2	y	0.11	$0.11y$
Total	\$9,700		\$863

Now let's set up the system. The total interest made in the one year was \$863 and the total invested was \$9,700, so we obtain the system

$$\begin{aligned} x+y &= 9700 \\ 0.07x+0.11y &= 863 \end{aligned}$$

At this point, we can solve using any method we choose. Since the coefficient of x and y in the first equation are both one, then let's use the method of substitution. We will solve for y in the first equation:

$$\begin{aligned} x+y &= 9700 \\ y &= 9700-x \end{aligned}$$

Now, we can substitute y into the second equation and solve:

$$\begin{aligned} 0.07x+0.11(9700-x) &= 863 && \text{Distribute} \\ 0.07x+1067-0.11x &= 863 && \text{Combine like terms} \\ -0.04x+1067 &= 863 && \text{Isolate the variable term} \\ -0.04x &= -204 && \text{Multiply by the reciprocal of } -0.04 \\ x &= 5100 && \text{Investment amount for Account 1} \end{aligned}$$

Since the investment amount for Account 1 was \$5,100, then the investment amount for Account 2 was \$4,600 ($9700-5100=4600$). Thus, the investment amounts for Account 1 and Account 2 was \$5,100 and \$4,600, respectively.

4.4.2 Mixture problems

Example 4.22

A farmer has two types of milk, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up with 42 gallons of 20% butterfat?

Solution.

First, we can make a table to organize the given information and then create a system. Let x represent the number of gallons of the 24% butterfat milk and y represent the number of gallons of the 18% butterfat milk.

	Amount	**Concentration**	**Total Butterfat**
24% butterfat	x	0.24	$0.24x$
18% butterfat	y	0.18	$0.18y$
20% butterfat	42	0.20	0.20(42)

Now let's set up the system:

$$\begin{aligned} x + y &= 42 \\ 0.24x + 0.18y &= 8.4 \end{aligned}$$

At this point, we can solve using any method we choose. Let's solve using elimination. We can choose to eliminate x and we will multiply the first equation by -0.24:

$$\begin{aligned} {\color{blue}-0.24} \cdot (x + y) &= (42) \cdot {\color{blue}-0.24} && \text{Distribute} \\ -0.24x - 0.24y &= -10.08 \end{aligned}$$

Notice the x variable terms have the same coefficients with opposite signs. Now we can add and eliminate x:

$$\begin{array}{rr} & -0.24x - 0.24y = -10.08 \\ + & 0.24x + 0.18y = 8.4 \\ \hline & -0.06y = -1.68 \end{array} \qquad \text{Add the equations}$$

Now, we can easily solve as usual:

$$\begin{aligned} -0.06y &= -1.68 && \text{Multiply by the reciprocal of } -0.06 \\ y &= 28 && \text{Number of gallons from the 18\% butterfat} \end{aligned}$$

Since the number of gallons from the 18% butterfat milk was 28, then the number of gallons from the 24% butterfat milk was 14 ($42 - 28 = 14$). Thus, the farmer will need 14 gallons of the 24% butterfat milk and 28 gallons of the 18% butterfat milk to make 42 gallons of a 20% butterfat milk.

Example 4.23

A solution of pure antifreeze is mixed with water to make a 65% antifreeze solution. How much of each should be used to make 70 liters?

Solution.

Mixture problems with a pure solution or water contains no other chemicals. For pure solutions, the percentage is 100% (or 1 in the table) and for water, the percentage is 0%. First, we can make a table to organize the given information and then create a system. Let a represent the number of liters of antifreeze and w represent the number of liters of water.

	Amount	**Concentration**	**Total Butterfat**
Antifreeze	a	1	$1a$
Water	w	0	0
65% Solution	70	0.65	0.65(70)

Now let's set up the system:

$$\begin{aligned} a + w &= 70 \\ 1a &= 45.5 \end{aligned}$$

At this point, we can solve using any method we choose. Since we see from the system that $a = 45.5$, then let's solve by substitution. We can put the second equation in for a in the first equation:

$$\begin{aligned} a + w &= 70 && \text{Plug-n-chug } a = 45.5 \\ 45.5 + w &= 70 && \text{Isolate the variable term} \\ w &= 24.5 && \text{The number of liters of water} \end{aligned}$$

Thus, the number of liters of water needed is 24.5 liters and antifreeze needed is 45.5 liters.

Example 4.24

In a candy shop, chocolate, which sells for \$4 per pound, is mixed with nuts, which are sold for \$2.50 per pound, to form a chocolate-nut candy which sells for \$3.50 a pound. How many pounds of each are used to make 30 pounds of the mixture?

Solution.

First, we can make a table to organize the given information and then create an equation. Let c represent the number of pounds of chocolate and n represent the number of pounds of nuts.

	Amount	**Cost**	**Total Cost**
Chocolate	c	\$4	$4c$
Nuts	n	\$2.50	$2.5n$
Mix	30	\$3.50	3.5(30)

Now let's set up the system:

$$\begin{aligned} c + n &= 30 \\ 4c + 2.5n &= 105 \end{aligned}$$

At this point, we can solve using any method we choose. Let's solve using elimination. We can choose to eliminate c and we will multiply the first equation by -4:

$$\begin{aligned} -4 \cdot (c + n) &= (30) \cdot -4 && \text{Distribute} \\ -4c - 4n &= -120 \end{aligned}$$

Notice the c variable terms have the same coefficients with opposite signs. Now we can add and eliminate c:

$$\begin{array}{rr} & -4c - 4n = -120 \\ + & 4c + 2.5n = 105 \\ \hline & -1.5n = -15 \end{array} \qquad \text{Add the equations}$$

Now, we can easily solve as usual:

$$\begin{aligned} -1.5n &= -15 && \text{Multiply by the reciprocal of } -1.5 \\ n &= 10 && \text{Number of pounds of nuts} \end{aligned}$$

Since we need 10 pounds of nuts, then this implies that we need 20 pounds of chocolate ($30-10 = 20$).

4.4.3 Uniform motion with unknown rates

When we looked at uniform motion in a prior chapter, we always were given something about the rate. However, now we discuss uniform motion with a wind force and a water current rates, where we know very little and will have to use a system to solve for the rates.

Example 4.25

Turkey the Pigeon travels the same distance of 72 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

Solution.

First, we can make a table to organize the given information and then create an equation. Let r represent the rate of Turkey in still air and w represent the rate of the wind. If Turkey travels with the wind, then Turkey is getting a little push from the wind, meaning traveling a little faster. If Turkey travels against the wind, then Turkey is getting a little push back from the wind, meaning traveling a little slower.

	rate	**time**	**distance**
with the wind	$r+w$	3	$3(r+w)$
against the wind	$r-w$	4	$4(r-w)$

Since Turkey is traveling a distance of 72 miles, then this is the distance for both routes. Now let's set up the system:

$$\begin{aligned} 3(r+w) &= 72 \\ 4(r-w) &= 72 \end{aligned}$$

Since 3 is a factor of 72 and 4 is a factor of 72, let's divide each side of each equation:

$$\begin{aligned} \frac{3(r+w)}{3} &= \frac{72}{3} \\ r+w &= 24 \end{aligned}$$

$$\begin{aligned} \frac{4(r-w)}{4} &= \frac{72}{4} \\ r-w &= 18 \end{aligned}$$

Notice the w variable terms have the same coefficients with opposite signs. Now we can add and eliminate w:

$$\begin{array}{rr} & r+w=24 \\ + & r-w=18 \\ \hline & 2r=42 \end{array} \qquad \text{Add the equations}$$

Now, we can easily solve as usual:

$$\begin{aligned} 2r &= 42 && \text{Multiply by the reciprocal of 2} \\ r &= 21 && \text{Rate of Turkey in still air} \end{aligned}$$

Since Turkey's rate is 21 miles per hour in still air, then this implies that the rate of the wind is 3 miles per hour $(24 - 21 = 3)$.

Example 4.26

A boat travels upstream for 156 miles in 3 hours and returns in 2 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Solution.

First, we can make a table to organize the given information and then create an equation. Let r represent the rate of boat in still water and c represent the rate of the current. If the boat travels with the current, then the boat is getting a little push from the current, meaning traveling a little faster. If the boat travels against the current, then the boat is getting a little push back from the current, meaning traveling a little slower.

	rate	**time**	**distance**
with the current	$r+c$	2	$2(r+c)$
against the current	$r-c$	3	$3(r-c)$

Since the boat is traveled a distance of 156 miles, then this is the distance for both routes. Now let's set up the system:

$$\begin{aligned} 2(r+c) &= 156 \\ 3(r-c) &= 156 \end{aligned}$$

Since 2 is a factor of 156 and 3 is a factor of 156, let's divide each side of each equation:

$$\frac{2(r+c)}{2} = \frac{156}{2}$$
$$r + c = 78$$

$$\frac{3(r-c)}{3} = \frac{156}{3}$$
$$r - c = 52$$

Notice the c variable terms have the same coefficients with opposite signs. Now we can add and eliminate c:

$$\begin{array}{rr} & r + c = 78 \\ + & r - c = 52 \\ \hline & 2r = 130 \end{array} \qquad \text{Add the equations}$$

Now, we can easily solve as usual:

$$\begin{aligned} 2r &= 130 && \text{Multiply by the reciprocal of 2} \\ r &= 65 && \text{Rate of the boat in still water} \end{aligned}$$

Since the rate of the boat in still water is 65 miles per hour, then this implies that the rate of the current is 13 miles per hour $(65 - 52 = 13)$.

4.4.4 Applications with Systems of Equations Homework

1. The attendance at a school concert was 578. Admission was \$2.00 for adults and \$1.50 for children. The total receipts were \$985.00. How many adults and how many children attended?

2. There were 429 people at a play. Admission was \$1 each for adults and 75 cents each for children. The receipts were \$372.50. How many children and how many adults attended?

3. There were 200 tickets sold for a women's basketball game. Tickets forstudents were 50 cents each and for adults 75 cents each. The total amount of money collected was \$132.50. How many of each type of ticket was sold?

4. There were 203 tickets sold for a volleyball game. For activity-card holders, the price was \$1.25 each and for non-card holders the price was \$2 each. The total amount of money collected was \$310. How many of each type of ticket was sold?

5. At a local ball game the hot dogs sold for \$2.50 each and the hamburgers sold for \$2.75 each. There were 131 total hamburgers and hot dogs sold for a total value of \$342. How many of each was sold?

6. At a recent Vikings game, there was \$445 in admission tickets. The cost of a student ticket was \$1.50 and the cost of a non-student ticket was \$2.50. A total of 232 tickets were sold. How many students and how many non-students attended the game?

7. A total of \$27,000 is invested, part of it at 12% and the rest at 13%. The total interest after one year is \$3,385. How much was invested at each rate?

8. A total of \$50,000 is invested, part of it at 5% and the rest at 7.5%. The total interest after one year is \$3,250. How much was invested at each rate?

9. A total of \$9,000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is \$1,030. How much was invested at each rate?

10. A total of \$18,000 is invested, part of it at 6% and the rest at 9%. The total interest after one year is \$1,248. How much was invested at each rate?

11. An inheritance of \$10,000 is invested in 2 ways, part at 9.5% and the remainder at 11%. The combined annual interest was \$1,038.50. How much was invested at each rate?

12. Kerry earned a total of \$900 last year on his investments. If \$7,000 was invested at a certain rate of return and \$9,000 was invested in a fund with a rate that was 2% higher, find the two rates of interest.

13. Jason earned \$256 interest last year on his investments. If \$1,600 was invested at a certain rate of return and \$2,400 was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

14. Millicent earned \$435 last year in interest. If \$3,000 was invested at a certain rate of return and \$4,500 was invested in a fund with a rate that was 2% lower, find the two rates of interest.

15. A total of \$8,500 is invested, part of it at 6% and the rest at 3.5%. The total interest after one year is \$385. How much was invested at each rate?

16. A total of \$12,000 was invested, part of it at 9% and the rest at 7.5%. The total interest after one year is \$1,005. How much was invested at each rate?

17. A total of \$15,000 is invested, part of it at 8% and the rest at 11%. The total interest after one year is \$1,455. How much was invested at each rate?

18. A total of \$17,500 is invested, part of it at 7.25% and the rest at 6.5%. The total interest after one year is \$1,227.50. How much was invested at each rate?

19. A total of \$6,000 is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is \$300. How much was invested at each rate?

20. A total of \$14,000 is invested, part of it at 5.5% and the rest at 9%. The total interest after one year is \$910. How much was invested at each rate?

21. A total of \$11,000 is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is \$797. How much was invested at each rate?

22. An investment portfolio earned \$2,010 in interest last year. If \$3,000 was invested at a certain rate of return and \$24,000 was invested in a fund with a rate that was 4% lower, find the two rates of interest.

23. Samantha earned \$1,480 in interest last year on her investments. If \$5,000 was invested at a certain rate of return and \$11,000 was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

24. Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100 cc. of a solution that is 68% acid?

25. A certain grade of milk contains 10% butterfat and a certain grade of cream 60% butterfat. How many quarts of each must be taken so as to obtain a mixture of 100 quarts that will be 45% butterfat?

26. A farmer has some cream which is 21% butterfat and some which is 15% butterfat. How many gallons of each must be mixed to produce 60 gallons of cream which is 19% butterfat?

27. A syrup manufacturer has some pure maple syrup and some which is 85% maple syrup. How many liters of each should be mixed to make 150liters which is 96% maple syrup?

28. A chemist wants to make 50 mL of a 16% acid solution by mixing a 13% acid solution and an 18% acid solution. How many milliliters of each solution should the chemist use?

29. A hair dye is made by blending 7% hydrogen peroxide solution and a 4% hydrogen peroxide solution. How many milliliters of each are used to make a 300 mL solution that is 5% hydrogen peroxide?

30. A paint that contains 21% green dye is mixed with a paint that contains 15% green dye. How many gallons of each must be used to make 60 gallons of paint that is 19% green dye?

31. A candy mix sells for \$2.20 per kilogram. It contains chocolates worth \$1.80 per kilogram and other candy worth \$3.00 per kilogram. How much of each are in 15 kilograms of the mixture?

32. To make a weed and feed mixture, the Green Thumb Garden Shop mixes fertilizer worth \$4 per pound with a weed killer worth \$8 per pound. The mixture will cost \$6.00 per pound. How much of each should be used to prepare 500 pounds of the mixture?

33. A grocer is mixing a 40-cent per pound coffee with a 60-cent per pound coffee to make a mixture worth 54¢ per pound How much of each kind of coffee should be used to make 70 pounds of the mixture?

34. A grocer wishes to mix sugar at 9 cents per pound with sugar at 6 cents per pound to make 60 pounds at 7 cents per pound. What quantity of each must he take?

35. A high-protein diet supplement that costs \$6.75 per pound is mixed with a vitamin supplement that costs \$3.25 per pound. How many pounds of each should be used to make 5 pounds of a mixture that costs \$4.65 per pound?

36. A goldsmith combined an alloy that costs \$4.30 per ounce with an alloy that costs \$1.80 per ounce. How many ounces of each were used to make a mixture of 200 ounces costing \$2.50 per ounce?

37. A grocery store offers a cheese and fruit sampler that combines cheddar cheese that costs \$8 per kilogram with kiwis that cost \$3 per kilogram. How many kilograms of each were used to make a 5 kilogram mixture that costs \$4.50 per kilogram?

38. A caterer made an ice cream punch by combining fruit juice that cost \$2.25 per gallon with ice cream that costs \$3.25 per gallon. How many gallons of each were used to make 100 gallons of punch costing \$2.50 per pound?

39. A clothing manufacturer has some pure silk thread and some thread that is 85% silk. How many kilograms of each must be woven together to make 75 kilograms of cloth that is 96% silk?

40. A carpet manufacturer blends two fibers, one 20% wool and the second 50% wool. How many pounds of each fiber should be woven together to produce 600 pounds of a fabric that is 28% wool?

41. The manager of a specialty food store combined almonds that cost \$4.50 per pound with walnuts that cost \$2.50 per pound. How many pounds of each were used to make a 100-pound mixture that cost \$3.24 per pound?

42. A tea that is 20% jasmine is blended with a tea that is 15% jasmine. How many pounds of each tea are used to make 5 pounds of tea that is 18% jasmine?

43. How many milliliters of pure chocolate must be added to 150 mL of chocolate topping that is 50% chocolate to make a topping that is 75% chocolate?

44. How many ounces of pure bran flakes must be added to 50 ounces of cereal that is 40% bran flakes to produce a mixture that is 50% bran flakes?

45. A ground meat mixture is formed by combining meat that costs \$2.20 per pound with meat that costs \$4.20 per pound. How many pounds of each were used to make a 50-pound mixture that costs \$3 per pound?

46. How many grams of pure water must be added to 50 grams of pure acid to make a solution that is 40% acid?

47. A lumber company combined oak wood chips that cost \$3.10 per pound with pine wood chips that cost \$2.50 per pound. How many pounds of each were used to make an 80-pound mixture costing \$2.65 per pound?

48. How many ounces of pure water must be added to 50 ounces of a 15% saline solution to make a saline solution that is 10% salt?

49. A boat travels upstream for 216 miles in 4 hours and returns in 3 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

50. A boat travels upstream for 12 miles in 3 hours and returns in 2 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

51. A boat travels upstream for 336 miles in 4 hours and returns in 3 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

52. Turkey the Pigeon travels the same distance of 280 miles in 5 hours against the wind as it does traveling 4 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

53. Turkey the Pigeon travels the same distance of 24 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

54. Turkey the Pigeon travels the same distance of 120 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

4.5 Systems of three linear equations in three variables

Solving systems of linear equations in three variables is very similar to the methods in which we solve linear systems in two variables. With linear systems in two variables, we reduced the system down to one linear equation in one variable (by substitution or addition). With linear systems in three variables, we apply the same method except we reduce the system down from three linear equations in three variables to two linear equations in two variables first, then to one linear equation in one variable.

Definition

A ***system of three linear equations in three variables*** is given in the form

$$\begin{cases} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= \ell \end{cases}$$

where $a, b, c, d, e, f, g, h, i, j, k$ and ℓ are coefficients and x, y, and z are variables. This system is represented in *standard form.*

4.5.1 Verifying solutions

As we did is the previous section for verifying solutions with two linear equations in two variables, we can verify whether an ***ordered triple*** is a solution to the system of linear equations in three variables.

Example 4.27

Determine whether $(5, -3, -4)$ is a solution to the system: $\begin{cases} x + y + z = -2 \\ x + 2y - 3z = 12 \\ 2x - 2y + z = -9 \end{cases}$

Solution.

To verify whether $(5, -3, -4)$ is the solution to the system, we plug-n-chug $(5, -3, -4)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for all equations in the system, then $(5, -3, -4)$ will be the solution to the system.

$$\begin{aligned} x + y + z &= -2 && \text{Plug-n-chug } x = 5, y = -3 \text{ and } z = -4 \\ 5 + (-3) + (-4) &\stackrel{?}{=} -2 && \text{Simplify} \\ 5 - 3 - 4 &\stackrel{?}{=} -2 && \text{Subtract} \\ -2 &= -2 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} x + 2y - 3z &= 12 && \text{Plug-n-chug } x = 5, y = -3 \text{ and } z = -4 \\ 5 + 2(-3) - 3(-4) &\stackrel{?}{=} 12 && \text{Simplify} \\ 11 &= 12 && \text{✗ False} \end{aligned}$$

Since the ordered triple $(5, -3, -4)$ makes the second equation false, then $(5, -3, -4)$ is not the solution to the system.

Example 4.28

Let's try a different ordered triple. Determine whether $(-1, 2, -3)$ is a solution to the system:

$$\begin{cases} x + y + z = -2 \\ x + 2y - 3z = 12 \\ 2x - 2y + z = -9 \end{cases}$$

Solution.

To verify whether $(-1, 2, -3)$ is the solution to the system, we plug-n-chug $(-1, 2, -3)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for all equations in the system, then $(-1, 2, -3)$ will be the solution to the system.

$$\begin{aligned} x + y + z &= -2 && \text{Plug-n-chug } x = -1, y = 2 \text{ and } z = -3 \\ -1 + 2 + (-3) &\stackrel{?}{=} -2 && \text{Simplify} \\ -2 &= -2 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} x + 2y - 3z &= 12 && \text{Plug-n-chug } x = -1, y = 2 \text{ and } z = -3 \\ -1 + 2(2) - 3(-3) &\stackrel{?}{=} 12 && \text{Simplify} \\ 12 &= 12 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the third equation:

$$\begin{aligned} 2x - 2y + z &= -9 && \text{Plug-n-chug } x = -1, y = 2 \text{ and } z = -3 \\ 2(-1) - 2(2) + (-3) &\stackrel{?}{=} -9 && \text{Simplify} \\ -9 &= -9 && \checkmark \text{ True} \end{aligned}$$

Since the ordered triple $(-1, 2, -3)$ makes all equations in the system true, then $(-1, 2, -3)$ is a solution to the system.

4.5.2 Solving systems of three linear equations in three variables

In Example 4.28, we determined that the ordered triple $(-1, 2, -3)$ is a solution to the system

$$\begin{cases} x + y + z = -2 \\ x + 2y - 3z = 12 \\ 2x - 2y + z = -9 \end{cases}$$

What does this mean? With systems of two linear equations in two variables, we know that if the system is consistent with an independent solution, then the solution is an ordered pair and two lines intersect. However, with three linear equations in three variables, what does a system that is consistent with an independent solution look like? We do know the answer is an ***ordered triple*** of form (x, y, z), but what does the graph look like? Well, let's take a look.

If we obtain a consistent system with an independent solution, then the solution is an ordered triple. This unique ordered triple represents the *intersection of three planes*.
However, experience from the previous sections tells us that this isn't always the case, right? With two linear equations in two variables, we have two special cases: *no solution* and *infinitely many solutions on line* $y = mx + b$.

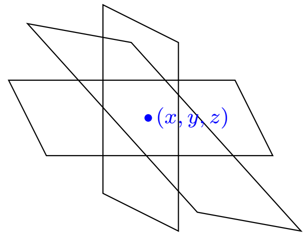
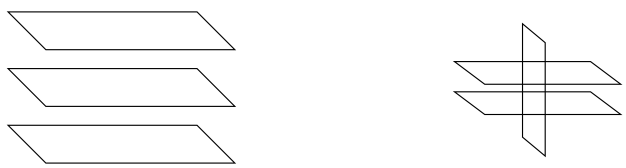


Figure 4.1: In this figure, we see the intersection of three planes at point (x, y, z). This consistent system is independent with solution (x, y, z).

Recall. No solution meant the two lines were parallel and never intersected. *Infinitely many solutions on line* $y = mx + b$ meant that the two lines were, in fact, the same line $y = mx + b$.

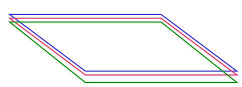
Let us take look at two cases with systems that are inconsistent, but now with three linear equations in three variable and with planes.



(a) Inconsistent system (b) Inconsistent system

Figure 4.2: Here we can see two examples of inconsistent systems where in 4.2a, all three planes could be parallel, and in 4.2b, one plane intersects two parallel planes.

Lastly, let us take look at a case with a consistent system that is dependent, but now with three linear equations in three variable and with planes.



(a) Consistent system that is dependent

Figure 4.3: In 4.3a, we see all three planes are the same plane and this example is called a consistent system that is dependent with solutions of form (x, y, z), where the solution is usually rewritten as $\{(x, y, z)|ax + by + cy = d\}$.

Example 4.29

Solve the system: $\begin{cases} 3x + 2y - z = -1 \\ -2x - 2y + 3z = 5 \\ 5x + 2y - z = 3 \end{cases}$

Solution.

Let's go ahead and number each equation so that we can identify each equation.

$$3x + 2y - z = -1 \tag{1}$$
$$-2x - 2y + 3z = 5 \tag{2}$$
$$5x + 2y - z = 3 \tag{3}$$

First, we choose a variable to eliminate. Then take two equations, say (1) and (2), and eliminate the chosen variable. Let's choose y since we can see that the coefficients of y are the same and we can easily eliminate it.

$$3x + 2y - z = -1 \tag{1}$$
$$-2x - 2y + 3z = 5 \tag{2}$$

Adding (1) and (2), we obtain

$$x + 2z = 4 \tag{4}$$

Now, let's take equations (2) and (3) and eliminate y again:

$$-2x - 2y + 3z = 5 \tag{2}$$
$$5x + 2y - z = 3 \tag{3}$$

Adding (2) and (3), we obtain

$$3x + 2z = 8 \tag{5}$$

Next, take equations (4) and (5). Notice, we have a system of two linear equations in two variables:

$$x + 2z = 4 \tag{4}$$
$$3x + 2z = 8 \tag{5}$$

We use the same process as we did in the previous section to obtain the solution for x and z. Then substitute those values into one of the original equations to obtain y. Thus, obtaining the ordered triple solution.

Let's choose to eliminate z and solve for x. First, let's multiply equation (4) by -1.

$$-1(x + 2z) = -1(4) \tag{4}$$
$$3x + 2z = 8 \tag{5}$$

Now, we can solve for x:

$$-x - 2z = -4 \tag{4}$$
$$3x + 2z = 8 \tag{5}$$

Adding these together, we get

$$2x = 4$$
$$x = 2$$

If $x = 2$, then this implies that $z = 1$, i.e.,

$$\begin{aligned} x + 2z &= 4 && (4) \\ 2 + 2z &= 4 \\ 2z &= 2 \\ z &= 1 \end{aligned}$$

Lastly, we take $x = 2$ and $z = 1$ and substitute them into one of the original equations, like (1), and solve for y:

$$\begin{aligned} 3x + 2y - z &= -1 && (1) \\ 3(2) + 2y - (1) &= -1 \\ 6 + 2y - 1 &= -1 \\ 5 + 2y &= -1 \\ 2y &= -6 \\ y &= -3 \end{aligned}$$

Thus, the point of intersection of the three planes is the ordered triple $(2, -3, 1)$.

World Note

Around 250 BCE, *The Nine Chapters on the Mathematical Art* were published in China. This book had 246 problems, and Chapter 8 was about solving systems of equations. One problem had four equations with five variables!

4.5.3 Special Cases

Looking back at Figure 1.2, we see that there are two special cases: Inconsistent systems and consistent systems with dependent solutions. Let's us see examples of these special cases.

Example 4.30

Solve the system: $\begin{cases} 5x - 4y + 3z = -4 \\ -10x + 8y - 6z = 8 \\ 15x - 12y + 9z = -12 \end{cases}$

Solution.

Let's go ahead and number each equation so that we can identify each equation.

$$\begin{aligned} 5x - 4y + 3z &= -4 && (1) \\ -10x + 8y - 6z &= 8 && (2) \\ 15x - 12y + 9z &= -12 && (3) \end{aligned}$$

First, we choose a variable to eliminate. Then take two equations, say (1) and (2), and eliminate the chosen variable. Let's choose z since we can see that the coefficients of z are almost the same and we can easily eliminate it. We first multiply equation (1) by 2 and then add.

$$\begin{aligned} 2(5x - 4y + 3z) &= 2(-4) && (1) \\ -10x + 8y - 6z &= 8 && (2) \end{aligned}$$

Now, we can add to eliminate z:

$$10x - 8y + 6z = -8 \quad (1)$$
$$-10x + 8y - 6z = 8 \quad (2)$$

Adding (1) and (2), we obtain

$$0 = 0 \quad \checkmark \text{ True}$$

Since all variables eliminate and we are left with a true statement, we know this is a consistent system with dependent solutions. However, what is the solution? We see from Figure 4.3a that this means all three planes are the same plane. Hence, we should write the solution as $\{(x, y, z)|5x - 4y + 3z = -4\}$.

Example 4.31

Solve the system: $\begin{cases} 3x - 4y + z = 2 \\ -9x + 12y - 3z = -5 \\ 4x - 2y - z = 3 \end{cases}$

Solution.

Let's go ahead and number each equation so that we can identify each equation.

$$3x - 4y + z = 2 \quad (1)$$
$$-9x + 12y - 3z = -5 \quad (2)$$
$$4x - 2y - z = 3 \quad (3)$$

First, we choose a variable to eliminate. Then take two equations, say (1) and (2), and eliminate the chosen variable. Let's choose x since we can see that the coefficients of x are almost the same and we can easily eliminate it. We first multiply equation (1) by 3 and then add.

$$3(3x - 4y + z) = 3(2) \quad (1)$$
$$-9x + 12y - 3z = -5 \quad (2)$$

Now, we can add to eliminate z:

$$9x - 12y + 3z = 6 \quad (1)$$
$$-9x + 12y - 3z = -5 \quad (2)$$

Adding (1) and (2), we obtain

$$0 = 1 \quad \text{✗ False}$$

Since all variables eliminate and we are left with a false statement, we know this is an inconsistent system and there is no solution.

Systems of equations in three (or more) variables are not any more difficult than systems of equations in two variables as long as we are careful to keep organized. It is possible to solve each system several different ways. We can use different pairs of equations or eliminate variables in a different order. In a future chapter, we have the opportunity to solve systems of nonlinear equations!

4.5.4 Systems of Three Linear Equations in Three Variables Homework

Solve each system. Determine if each system is consistent, independent or dependent, or inconsistent. Write the solution as an ordered triple, if possible.

1. $a - 2b + c = 5$
$2a + b - c = -1$
$3a + 3b - 2c = -4$

2. $3x + y - z = 11$
$x + 3y = z + 13$
$x + y - 3z = 11$

3. $x + 6y + 3z = 4$
$2x + y + 2z = 3$
$3x - 2y + z = 0$

4. $x + y + z = 6$
$2x - y - z = -3$
$x - 2y + 3z = 6$

5. $x + y - z = 0$
$x - y - z = 0$
$x + y + 2z = 0$

6. $-2x + y - 3z = 1$
$x - 4y + z = 6$
$4x + 16y + 4z = 24$

7. $2x + y - 3z = 0$
$x - 4y + z = 0$
$4x + 16y + 4z = 0$

8. $3x + 2y + 2z = 3$
$x + 2y - z = 5$
$2x - 4y + z = 0$

9. $x - 2y + 3z = 4$
$2x - y + z = -1$
$4x + y + z = 1$

10. $x - y + 2z = 0$
$x - 2y + 3z = -1$
$2x - 2y + z = -3$

11. $4x - 3y + 2z = 40$
$5x + 9y - 7z = 47$
$9x + 8y - 3z = 97$

12. $2x + 3y = z - 1$
$3x = 8z - 1$
$5y + 7z = -1$

13. $x + y + z = 2$
$6x - 4y + 5z = 31$
$5x + 2y + 2z = 13$

14. $x - y + 2z = -3$
$x + 2y + 3z = 4$
$2x + y + z = -3$

15. $x + y - z = 0$
$x + 2y - 4z = 0$
$2x + y + z = 0$

16. $x + 2y - z = 4$
$4x - 3y + z = 8$
$5x - y = 12$

17. $4x + 12y + 16z = 4$
$3x + 4y + 5z = 3$
$x + 8y + 11z = 1$

18. $4x + 12y + 16z = 0$
$3x + 4y + 5z = 0$
$x + 8y + 11z = 0$

19. $p + q + r = 1$
$p + 2q + 3r = 4$
$4p + 5q + 6r = 7$

20. $x + 2y - 3z = 9$
$2x - y + 2z = -8$
$3x - y - 4z = 3$

21. $4x - 7y + 3z = 1$
$3x + y - 2z = 4$
$4x - 7y + 3z = 6$

22. $3x + y - z = 10$
$8x - y - 6z = -3$
$5x - 2y - 5z = 1$

23. $3x + 3y - 2z = 13$
$6x + 2y - 5z = 13$
$5x - 2y - 5z = -1$

24. $3x - 4y + 2z = 1$
$2x + 3y - 3z = -1$
$x + 10y - 8z = 7$

25. $m + 6n + 3p = 8$
$3m + 4n = -3$
$5m + 7n = 1$

26. $2x - 3y + 5z = 1$
$3x + 2y - z = 4$
$4x + 7y - 7z = 7$

27. $2x + y = z$
$4x + z = 4y$
$y = x + 1$

28. $3x + 2y = z + 2$
$y = 1 - 2x$
$3z = -2y$

29. $-2w + 2x + 2y - 2z = -10$
$w + x + y + z = -5$
$3w + 2x + 2y + 4z = -11$
$w + 3x - 2y + 2z = -6$

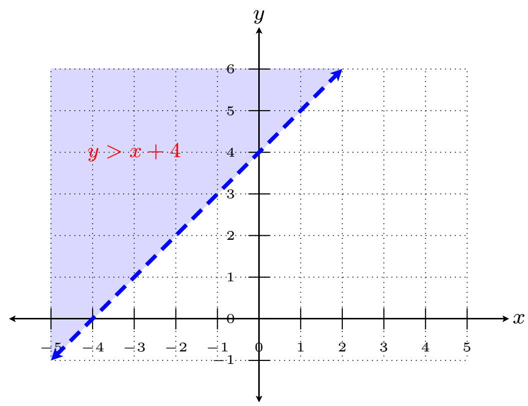
30. $w + x + y + z = 2$
$w + 2x + 2y + 4z = 1$
$-w + x - y - z = -6$
$-w + 3x + y - z = -2$

31. $-w + 2x - 3y + z = -8$
$-w + x + y - z = -4$
$w + x + y + z = 22$
$-w + x - y - z = -14$

32. $w + x - y + z = 0$
$-w + 2x + 2y + z = 5$
$-w + 3x + y - z = -4$
$-2w + x + y - 3z = -7$

4.6 Systems of two linear inequalities in two variables

In a previous section, we discussed linear inequalities in two variables, where we have the boundary line, dashed or solid, and shading either above or below the y-intercept, depending on the inequality symbol. For example, let's recall $y > x + 4$ and its graph.



Well, let's use this same idea for finding the solution to a system of two linear inequalities in two variables. The process is similar, except for the fact that the solution, in this case, is the region with overlap shading. We can present the solution as a shaded region.

Definition

A ***system of two linear inequalities in two variables*** is a system of the form

$$\begin{cases} ax + by & < c \\ dx + ey & < f \end{cases}$$

where $a, b, c, d, e,$ and f are coefficients, x and y are variables, and the form is the same for any combination of $>, \leq, \geq$. This system is represented in *standard form.*

4.6.1 Verifying solutions

Example 4.32

Determine whether the ordered-pairs are a solutions to the system: $\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$

a) $(-2, 4)$ **b)** $(3, 1)$

Solution.

We substitute the ordered pairs into the inequalities and determine if the results are true for both inequalities.

a) Let's substitute $(-2, 4)$ into the inequalities and determine if the ordered-pair is a solution to both inequalities.

$$\begin{aligned} x + 4y &\stackrel{?}{\geq} 10 && \text{Substitute } x = -2 \text{ and } y = 4 \\ -2 + 4(4) &\stackrel{?}{\geq} 10 && \text{Simplify} \\ 14 &\geq 10 && \checkmark \text{ True} \end{aligned}$$

$$\begin{aligned} 3x - 2y &\stackrel{?}{<} 12 && \text{Substitute } x = -2 \text{ and } y = 4 \\ 3(-2) - 2(4) &\stackrel{?}{<} 12 && \text{Simplify} \\ -14 &< 12 && \checkmark \text{ True} \end{aligned}$$

Hence, $(-2, 4)$ is a solution to the system.

b) Let's substitute $(3, 1)$ into the inequality and determine if the ordered-pair is a solution to both inequalities.

$$\begin{aligned} x + 4y &\stackrel{?}{\geq} 10 && \text{Substitute } x = 3 \text{ and } y = 1 \\ 3 + 4(1) &\stackrel{?}{\geq} 10 && \text{Simplify} \\ 7 &\ngeq 10 && \text{✗ False} \end{aligned}$$

Hence, $(3, 1)$ is not a solution to the system since $(3, 1)$ is not a solution to the first inequality $x + 4y \geq 10$.

4.6.2 Solving a system of two linear inequalities in two variables by graphing

In this section, we solve a system of of two linear inequalities in two variables by graphing. As we can see in the steps given, the process is similar to when we graphed linear inequalities in two variables.

We have to ask, "What is a solution to a system of two linear inequalities in two variables?" Recall, in previous sections of this chapter, we said the solution to a consistent system that is independent was a point of intersection. The intersection point still plays a role with these systems, but we are not looking for one point. We need *all* points in a shaded common area that make both inequalities in the system true. The intersection point is the corner for this common shaded region.

Steps for graphing a system of linear inequalities in two variables

Given a system of two linear inequalities in two variables,

$$\begin{cases} ax + by & < c \\ dx + ey & < f \end{cases},$$

we use the steps below to graph the system, where the same process is applied for any combination of $>, \leq, \geq$.

Step 1. Rewrite the inequalities in slope-intercept form, i.e., $y = mx + b$.

Step 2. Graph the boundary lines according to the two cases:

Case 1. If the inequality is $<$ or $>$, then the boundary line is dashed.

Case 2. If the inequality is $\geq$ or $\leq$, then the boundary line is solid.

Step 3. Shade the appropriate regions for each linear inequality.

Step 4. The solution is the region where the shading overlaps.

Step 5. Verify the solution by taking a test point where the shading overlaps that is *not on the boundary lines.* **Ask:** *Does this ordered pair make both inequalities true?*

Example 4.33

Solve the system by graphing: $\begin{cases} 3x - 2y \leq 6 \\ y > -\dfrac{1}{4}x + 5 \end{cases}$

Solution.

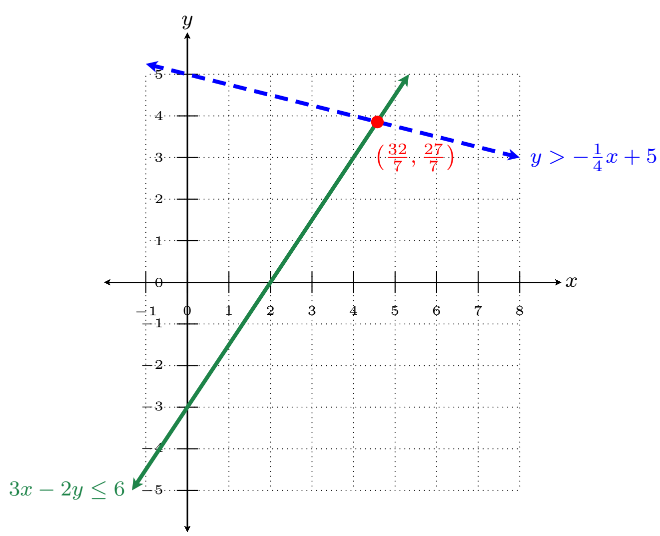
Let's follow the steps given above to graph the system of linear inequalities in two variables.

Step 1. Rewrite the inequalities in slope-intercept form, i.e., $y = mx + b$.

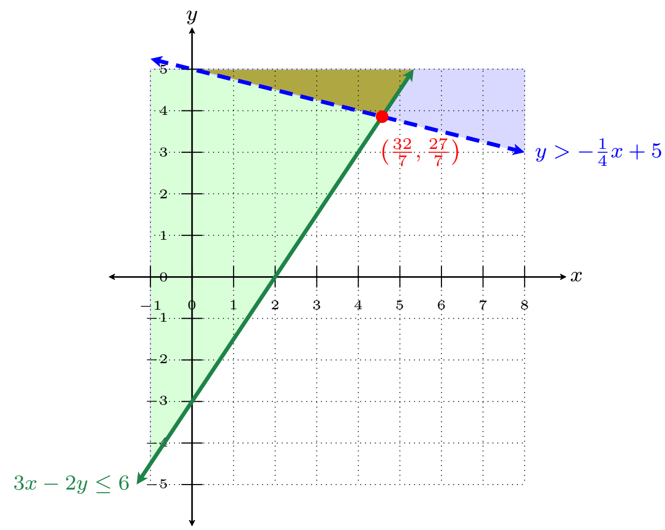
$$\begin{aligned} 3x - 2y &\leq 6 \\ -2y &\leq -3x + 6 \\ y &\geq \frac{3}{2}x - 3 \end{aligned}$$

The second inequality is already in slope-intercept form, i.e., $y > -\dfrac{1}{4}x + 5$.

Step 2. Graph the boundary lines according to the two cases. Since we have $\geq$ and $>$, then we have both cases where we have a solid line with $\geq$ and a dashed line with $>$.



Step 3. Shade the appropriate regions for each linear inequality.



Step 4. The solution is the region where the shading overlaps.

The region where the shading overlaps is the darkened region in the figure above (or olive green). Notice the point of intersection at $\left(\frac{32}{7}, \frac{27}{7}\right)$. The point of intersection is still critical in finding the solution to these types of systems.

Step 5. Verify the solution by taking a test point where the shading overlaps that is *not on the boundary lines*. Let's pick the test point $(4, 5)$ as it is a great choice!

$$\begin{aligned} 3x - 2y &\stackrel{?}{\leq} 6 \\ 3(4) - 2(5) &\stackrel{?}{\leq} 6 \\ 2 &\leq 6 \quad \checkmark \text{ True} \end{aligned}$$

$$\begin{aligned} y &\stackrel{?}{>} -\frac{1}{4}x + 5 \\ 5 &\stackrel{?}{>} -\frac{1}{4}(4) + 5 \\ 5 &> 4 \quad \checkmark \text{ True} \end{aligned}$$

Hence, $(4, 5)$ makes both inequalities true.

Thus, the darkened region (or olive green) is the solution.

4.6.3 Systems of Two Linear Inequalities in Two Variables Homework

Determine whether the given ordered pairs are solutions to the given system.

1. $y < \frac{1}{4}x + 2$
$y \geq -4x + 1$

$(6, 2)$, $(-4, 2)$, $(9, 1)$

2. $y \leq x + 1$
$y > -\frac{1}{4}x - 2$

$(-5, 1)$, $(10, 1)$, $(4, -2)$

3. $y < 2x - 1$
$y \geq -x - 2$

$(2, -1)$; $(-1, -1)$, $(-5, -2)$

Solve each system by graphing.

4. $y > \frac{1}{4}x - 1$
$y < -\frac{1}{2}x + 1$

5. $y < 2x + 1$
$y > -\frac{1}{2}x + 2$

6. $y < 4x - 2$
$y > -2x - 1$

7. $y > \frac{1}{2}x - 2$
$y < -\frac{1}{4}x + 2$

8. $x \leq 6$
$y < 4$

9. $x > -2$
$y > 4$

10. $x \leq 3$
$y > -2x + 5$

11. $x \leq 2$
$y > -4x + 5$

12. $y < 2x + 2$
$y > -4x - 1$

13. $y > 2x - 2$
$y < -x + 2$

14. $2x + y \geq 6$
$2x + 5y \leq 10$

15. $-2x - y \geq 6$
$-2x - 3y \leq 6$

16. $-2x + y \geq -6$
$-2x + 5y \leq -10$

17. $2x - y \leq -4$
$-x + 5y \geq -5$

4.7 Systems of Equations: Answers to the Homework Exercises

Graphing

1. No

3. Yes

5. $(-1, 2)$

7. No Solution

9. $(-3, -1)$

11. $(1, -3)$

13. $(2, -2)$

15. $(3, 2)$

17. $(-1, -2)$

19. $(4, -2)$

21. $(-3, 1)$

23. $(4, 4)$

25. $(4, -4)$

27. No Solution

29. $(2, -1)$

31. $(2, 3)$

33. $(-3, 1)$

The Substitution Method

1. $(1, -3)$; consistent; independent

3. $(-1, -2)$; consistent; independent

5. $(3, 3)$; consistent; independent

7. $(-2, -6)$; consistent; independent

9. $(-1, 8)$; consistent; independent

11. $(1, 7)$; consistent; independent

13. $(1, 3)$; consistent; independent

15. $(4, -3)$; consistent; independent

17. $(0, 3)$; consistent; independent

19. $(-4, -1)$; consistent; independent

21. $(-3, 3)$; consistent; independent

23. $(-1, 0)$; consistent; independent

25. $(8, -8)$; consistent; independent

27. $(1, -3)$; consistent; independent

29. $(-4, 3)$; consistent; independent

31. $(0, 2)$; consistent; independent

33. $(4, -2)$; consistent; independent

35. $(-4, 8)$; consistent; independent

37. $(1, -4)$; consistent; independent

39. $(2, 5)$; consistent; independent

The Addition Method

1. $(-2, 4)$; consistent; independent

3. No solution; inconsistent

5. $(-3, -5)$; consistent; independent

7. $(0, 4)$; consistent; independent

9. $(4, 6)$; consistent; independent

11. $(0, -4)$; consistent; independent

13. $(-1, -2)$; consistent; independent

15. $(-8, 9)$; consistent; independent

17. $(0, 0)$; consistent; independent

19. Infinite number of solutions; consistent; dependent

21. $(2, -2)$; consistent; independent

23. $(1, -2)$; consistent; independent

25. $(0, 3)$; consistent; independent

27. $(1, 2)$; consistent; independent

29. $(2, -2)$; consistent; independent

31. $(-3, 0)$; consistent; independent

33. $(-1, 1)$; consistent; independent

Applications with Systems of Equations

1. 236 adult, 342 child

3. 130 adults, 70 students

5. 73 hotdogs, 58 hamburgers

7. \$12,500 @ 12%; \$14,500 @ 13%

9. \$2,500 @ 10%; \$6,500 @ 12%

11. \$4,100 @ 9.5%; \$5,900 @ 11%

13. \$1,600 @ 4%; \$2,400 @ 8%

15. \$3,500 @ 6%; \$5,000 @ 3.5%

17. \$6,500 @ 8%; \$8,500 @ 11%

19. \$3,000 @ 4.25%; \$3,000 @ 5.75%

21. \$7,500 @ 6.8%; \$3,500 @ 8.2%

23. \$5,000 @ 12%; \$11,000 @ 8%

25. 30, 70

27. 40, 110

29. 100, 200

31. 10, 5

33. 21, 49

35. 2, 3

37. 1.5, 3.5

39. 55, 20

41. 35, 63

43. 1.2

45. 10

47. 75

49. 63 mph in still water; 9 mph current's rate

51. 5 mph in still water; 1 mph current's rate

53. $\frac{783}{8}$ mph in still air; $\frac{113}{8}$ mph wind's rate

55. 35 mph in still air; 5 mph wind's rate

Systems of Three Linear Equations in Three Variables

1. $(1,-1,2)$; consistent; independent

3. $(-2,-1,4)$; consistent; independent

5. $(0,0,0)$; consistent; independent

7. $(0,0,0)$; consistent; independent

9. $(1,2,3)$; consistent; independent

11. $(10,2,3)$; consistent; independent

13. $(3,-2,1)$; consistent; independent

15. $\{(x,y,z)|x+y-z=0\}$; consistent; dependent

17. $\{(x,y,z)|4x+12y+16z=4\}$; consistent; dependent

19. $\{(p,q,r)|p+q+r=1\}$; consistent; dependent

21. no solution; inconsistent

23. $(2,3,1)$; consistent; independent

25. $(-25,18,-25)$; consistent; independent

27. $(1,2,4)$; consistent; independent

29. $(1,-3,-2,-1)$; consistent; independent

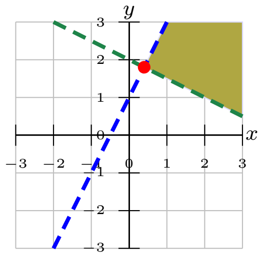
31. $(7,4,5,6)$; consistent; independent

Systems of Two Linear Inequalities in Two Variables

1. yes, no, yes

3. yes, no, no

5.



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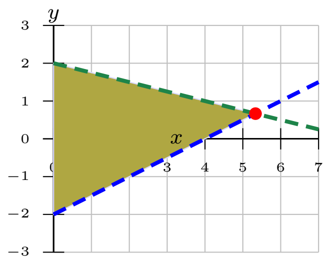

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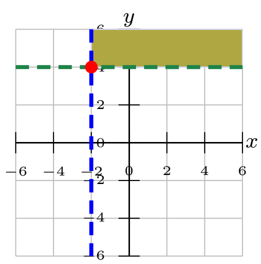

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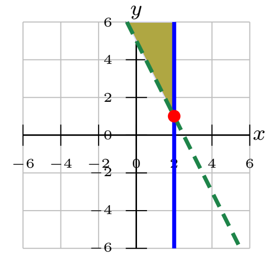

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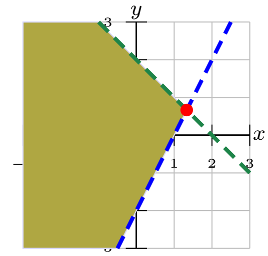

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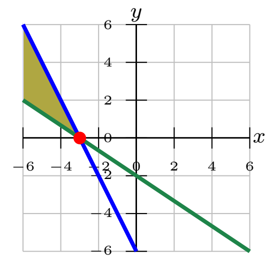
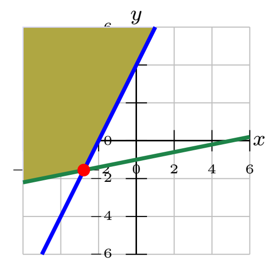


17.



Chapter 5

Functions

Chapter Objectives

By the end of this chapter, the student should be able to

- Evaluate and define a function
- Identify the independent and dependent variable and their units
- Apply algebraic operations on functions
- Recognize the shape of a function's graph with its name and formula

There are many different types of equations that we can work with in algebra because an equation gives the relationship between a variable(s) and numbers. For example,

$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{4} = 1 \quad \text{or} \quad y = x^2 - 2x + 7 \quad \text{or} \quad \sqrt{y+x} - 7 = xy$$

all give relationships between variables and numbers. Some of these relationships are called ***functions***.

Definition

A ***function*** is when one input of a relation is linked to only one output of the relation, i.e., a ***function*** has only one y for one x.

Function notation is represented by $f(x)$ such that

$$f(x) = y,$$

and we say f *is a function of* x.

5.1 Introduction to functions

A great way to visualize the definition of a function is to look at the graphs of a few relationships.

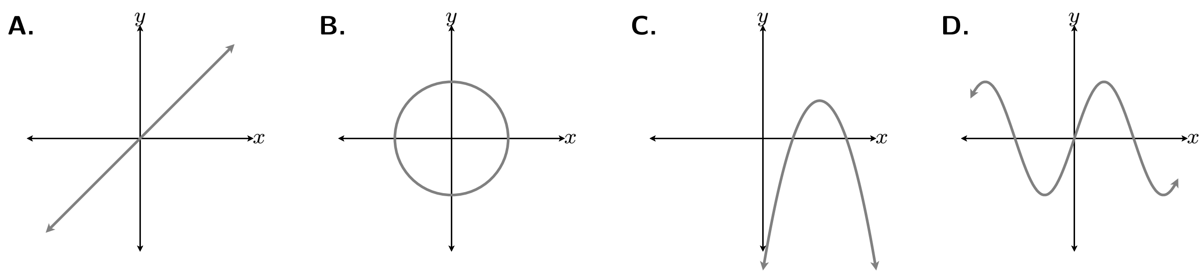
5.1.1 Vertical line test

Note

A given graph is a function if and only if every vertical line intersects the graph at most once, i.e., for every possible drawn vertical line through the graph, the line only intersects the graph at most one time. This test is called the ***vertical line test***.

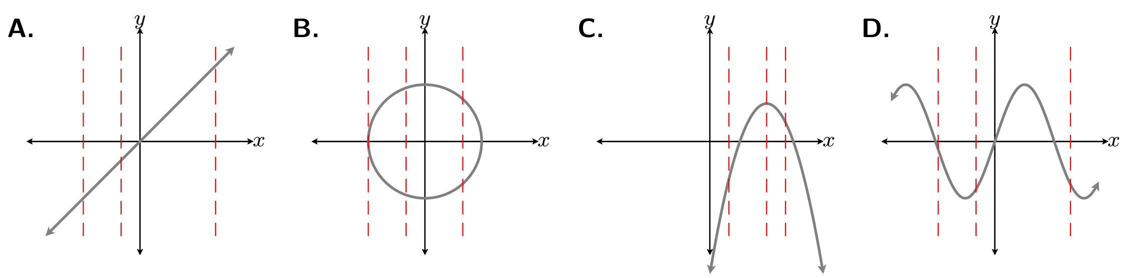
Example 5.1

Determine which of the following graphs represents a function.



Solution.

First, we draw vertical lines through each of these graphs. Then we determine the number of times the lines intersect the graph for each graph. If the vertical lines only intersect each graph at most once, then the graph is a function per the vertical line test.



We can see for all graphs, except for **B.**, pass the vertical line test. Notice on graphs **A.**, **C.**, and **D.**, all vertical lines pass through the graphs only once. However, in **B.**, we see the line intersects the graph two times. Even though the far left line touches the circle only once, the vertical line test fails for the other lines. Hence, the vertical line test fails for **B.** and **B.** is not a function, where graphs **A.**, **C.**, and **D.** represent functions.

5.1.2 Independent and dependent variables

Definition

We call $f(x)$ or y the ***dependent variable*** and x the ***independent variable***. Hence, the independent variable is also known as the *input* and the dependent variable is also known as the *output*.

Example 5.2

What is the independent and dependent variable?

a) $f(x) = \dfrac{3}{2}x + 1$

Solution.

Since the variable inside the parenthesis of $f(x)$ is x, then the independent variable is x, and $f(x)$ is the dependent variable.

b) $g(r) = r^3$

Solution.

Since the variable inside the parenthesis of $g(r)$ is r, then the independent variable is r, and $g(r)$ is the dependent variable.

c) $h(t) = \dfrac{t}{t^2 - 25}$

Solution.

Since the variable inside the parenthesis of $h(t)$ is t, then the independent variable is t, and $h(t)$ is the dependent variable.

d) $y = \sqrt{n - 16}$

Solution.

Since the variable on the right side is n, then the independent variable is n, and y is the dependent variable. Note, we could rewrite y so that it is clear that n is the independent variable by writing $y(n)$. This is good example of the reasons we like function notation- to make the independent and dependent variables more obvious.

Example 5.3

What is the independent and dependent variable? What does each variable represent?

a) The cost $C(x)$, where x is the number of miles driven, of renting a car for a day is $C(x) = 1.46x + 25$.

Solution.

Since the variable inside the parenthesis of $C(x)$ is x, then the independent variable is x, where x represents the number of miles driven, and $C(x)$ is the dependent variable, where $C(x)$ represents the cost of renting a car.

b) A rocket is launched at $t = 0$ seconds. Its height, in feet, above sea-level, as a function of time, t, is given by $h(t) = -16t^2 + 96t + 256$.

Solution.

Since the variable inside the parenthesis of $h(t)$ is t, then the independent variable is t, where t is the number of seconds, and $h(t)$ is the dependent variable, where $h(t)$ represents the height in feet after t seconds.

c) The profit for a certain commodity, n, where n is in units, is given by the function $P(n) = -25n^2 + 425n + 1500$.

Solution.

Since the variable inside the parenthesis of $P(n)$ is n, then the independent variable is n, where n is the number of units, and $P(n)$ is the dependent variable, where $P(n)$ represents the profit for selling n units of a commodity.

d) The revenue, $R(x)$, of producing and selling x Awesome Hearing Aids is modeled by the function $R(x) = -6x^2 + 67x$.

Solution.

Since the variable inside the parenthesis of $R(x)$ is x, then the independent variable is x, where x is the number of hearing aids, and $R(x)$ is the dependent variable, where $R(x)$ represents the revenue after selling x hearing aids.

World Note

The concept of a function was first introduced by Arab mathematician Sharaf al-Din al-Tusi in the late 12^{th} century.

5.1.3 Domains of functions

Once we know a relationship is a function, we may be interested in all the values that we can plug-n-chug into the function. The set of all values we are allowed to plug-n-chug into a function is called the ***domain***.

Definition

The ***domain*** of a function is the set of all inputs of the relation, i.e., all that x can be. The ***range*** of a function is the set of all outputs of the relation, i.e., all that y can be.

Note

When finding the domain, it is more efficient to consider the values that should be excluded from the domain and then exclude those values from the set.

Example 5.4

Find the domain: $f(x) = \dfrac{3x-1}{x^2+x-6}$

Solution.

Taking the above note into consideration, let's try to find the x values that should be excluded from the domain. We know, with fractions, that the denominator cannot be zero. Hence, we know we should exclude any values for x that make the denominator zero. Let's find these x values by setting the denominator to zero and solve.

$$\begin{aligned} x^2+x-6&=0 && \text{Solve by factoring}\\ (x+3)(x-2)&=0 && \text{Set each factor equal to zero}\\ x+3=0 \quad \text{or} \quad x-2&=0 && \text{Solve each equation}\\ x=-3 \quad \text{or} \quad x&=2 && x \text{ values that should be excluded from the domain} \end{aligned}$$

This means that x can be any value except for -3 and 2. If x were one of these two values, the function would be *undefined*. Thus, the domain for $f(x)$ is $\{x|x \neq -3, 2\}$ or, in interval notation, $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

Example 5.5

Find the domain: $g(x) = 3x^2 - x$

Solution.

Since there are no obvious characteristics of $g(x)$ such that $g(x)$ would contain any excluded values, then we say $g(x)$ has domain of *all real numbers* or, in interval notation, $(-\infty, \infty)$. In a future section, we will look at the graphs of functions, which may help us in finding the domain, too.

Example 5.6

Find the domain: $x(t) = \sqrt{2t-3}$

Solution.

Taking the above note into consideration, let's try to find the t values that should be excluded from the domain. We know, with square roots, that the radicand cannot be less than zero. Hence, we know we should exclude any values for t that make the radicand less than zero. Let's find these t values by setting the radicand to less than zero.

$$\begin{aligned} 2t-3&<0 && \text{Solve for the inequality}\\ 2t&<3 && \text{Divide by the coefficient of } t\\ t&<\frac{3}{2} && t \text{ values that should be excluded from the domain} \end{aligned}$$

This means that t can be any value except for any numbers less than $\frac{3}{2}$. If t were a value less than $\frac{3}{2}$, then the function would be *undefined*. Thus, the domain for $x(t)$ is $\left\{t \middle| t \geq \frac{3}{2}\right\}$ or, in interval notation, $\left(\frac{3}{2}, \infty\right)$.

5.1.4 Function notation

Let's take functions further by one step. Since functions are relationships between input and output values, and are represented by graphs, then we certainly can evaluate functions for certain input and output values.

Example 5.7

Evaluate $f(x) = 3x^2 - 4x$ for $x = -2$. What is the ordered-pair?

Solution.

$$\begin{aligned}
f(x) &= 3x^2 - 4x && \text{Substitute } -2 \text{ for every } x \text{ in the function}\\
f(-2) &= 3(-2)^2 - 4(-2) && \text{Simplify}\\
f(-2) &= 20 && \text{When the input is } -2\text{, the output is } 20
\end{aligned}$$

Hence, $f(-2) = 20$. Thus, the ordered-pair, (x, y), is $(-2, 20)$.

Example 5.8

Given $h(x) = 3^{2x-6}$, find $h(4)$. What is the ordered-pair?

Solution.

$$\begin{aligned}
h(x) &= 3^{2x-6} && \text{Substitute } 4 \text{ for every } x \text{ in the function}\\
h(4) &= 3^{2(4)-6} && \text{Simplify}\\
h(4) &= 3^{8-6} && \text{Subtract in the exponent}\\
h(4) &= 3^2 && \text{Evaluate}\\
h(4) &= 9 && \text{When the input is } 4\text{, the output is } 9
\end{aligned}$$

Hence, $h(4) = 9$. Thus, the ordered-pair, (x, y), is $(4, 9)$.

Example 5.9

Given $k(a) = 2|a + 4|$, find $k(-7)$. What is the ordered-pair?

Solution.

$$\begin{aligned}
k(a) &= 2|a + 4| && \text{Substitute } -7 \text{ for every } a \text{ in the function}\\
k(-7) &= 2|-7 + 4| && \text{Simplfy}\\
k(-7) &= 6 && \text{When the input is } -7\text{, the output is } 6
\end{aligned}$$

Hence, $k(-7) = 6$. Thus, the ordered-pair, (x, y), is $(-7, 6)$.

5.1.5 Evaluate functions with expressions

As the above examples show, the function can take many different forms, but the method to evaluate the function never changes, just the problems do. This leads into substituting expressions into functions using the same process.

Example 5.10

Given $g(x) = x^4 + 1$, find $g(3x)$.

Solution.

First, recall the method for evaluating functions never changes, just the problems. Hence, we will substitute $3x$ for every x in g, as usual.

$$\begin{aligned} g(x) &= x^4+1 && \text{Substitute } 3x \text{ for every } x \text{ in the function} \\ g(3x) &= (3x)^4+1 && \text{Simplify} \\ g(3x) &= 81x^4+1 && \text{When the input is } 3x\text{, the output is } 81x^4+1 \end{aligned}$$

Example 5.11

Given $p(t)=t^2-t$, find $p(t+1)$.

Solution.

First, recall the method for evaluating functions never changes, just the problems. Hence, we will substitute $t+1$ for every t in p, as usual.

$$\begin{aligned} p(t) &= t^2-t && \text{Substitute } t+1 \text{ for every } t \text{ in the function} \\ p(t+1) &= (t+1)^2-(t+1) && \text{Simplify by squaring the first term} \\ p(t+1) &= t^2+2t+1-(t+1) && \text{Distribute the negative} \\ p(t+1) &= t^2+2t+1-t-1 && \text{Simplify} \\ p(t+1) &= t^2+t && \text{When the input is } t+1\text{, the output is } t^2+t \end{aligned}$$

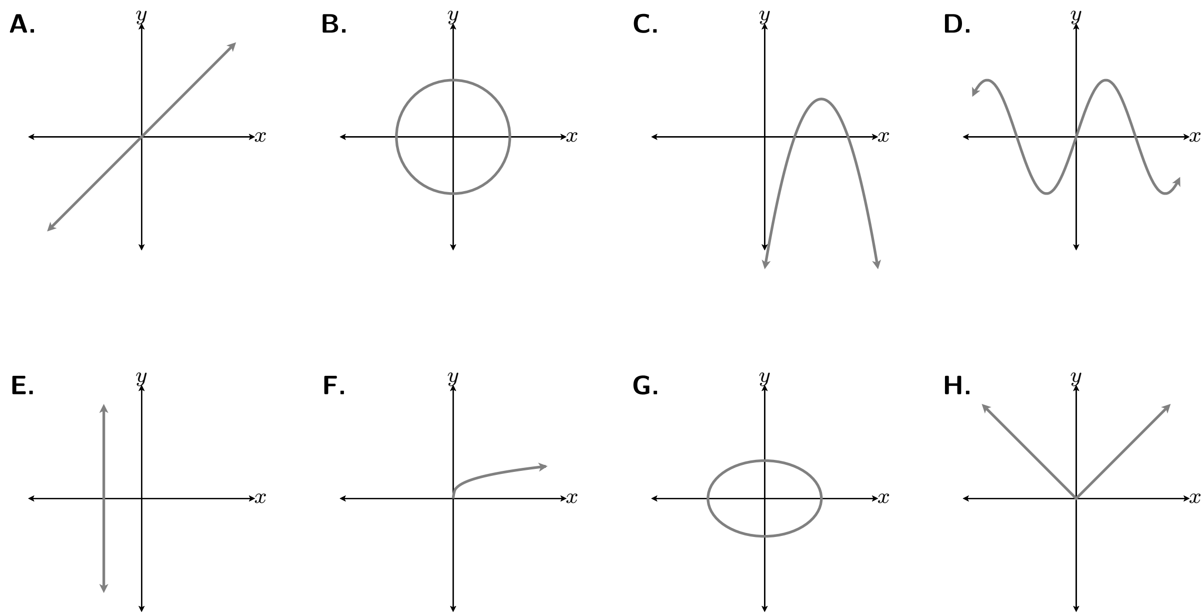
Note

In these last examples, essentially we are substituting a function into another function. The interesting part about these examples is when we substitute an expression into a function, we are creating an entirely new function. How awesome, right?

5.1.6 Introduction to Functions Homework

Determine whether the given ordered pair(s) is a solution to the system.

1. Determine which of the following are functions. If not, explain.



Specify the domain of each following function. List the independent and dependent variables, and their units.

2. $f(x) = -5x + 1$

3. $s(t) = \dfrac{1}{t^2}$

4. $s(t) = \dfrac{1}{t^2 + 1}$

5. $f(x) = \dfrac{-2}{x^2 - 3x - 4}$

6. $y(x) = \dfrac{x}{x^2 - 25}$

7. $f(x) = \sqrt{5 - 4x}$

8. $f(x) = x^2 - 3x - 4$

9. $f(x) = \sqrt{x - 16}$

10. $h(x) = \dfrac{\sqrt{3x - 12}}{x^2 - 25}$

11. A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h(t) = -4.9t^2 + 190t + 423$.

12. The profit (in dollars), P, for a certain commodity,n, where n is in units, is given by the function $P(n) = -25n^2 + 375n + 850$.

13. The revenue in dollars, $R(x)$, of producing and selling x Awesome Hearing Aids is modeled by the function $R(x) = x^2 + 55x$.

14. The cost in dollars $C(x)$, where x is the number of miles driven, of renting a car for a day is $C(x) = 1.25x + 33$.

For each given function, evaluate the function at the value or expression. What is the ordered-pair?

15. $g(x) = 4x - 4;\ g(0)$

16. $f(x) = |3x + 1| + 1;\ f(0)$

17. $f(n) = -2| - n - 2| + 1;\ f(-6)$

18. $f(t) = 3^t - 2;\ f(-2)$

19. $f(t) = |t + 3|;\ f(10)$

20. $w(n) = 4n + 3;\ w(2)$

21. $w(n) = 2^n + 2;\ w(-2)$

22. $p(n) = -3|n|;\ p(7)$

23. $p(t) = -t^3 + t;\ p(4)$

24. $k(n) = |n - 1|;\ k(3)$

25. $g(n) = -3 \cdot 5^{-n};\ g(2)$

26. $f(x) = x^2 + 4;\ f(-9)$

27. $f(n) = n - 3;\ f(10)$

28. $f(a) = 3^{a-1} - 3;\ f(2)$

29. $w(x) = x^2 + 4x;\ w(-5)$

30. $w(x) = -4x + 3;\ w(6)$

31. $p(x) = -|x| + 1;\ p(5)$

32. $k(a) = a + 3;\ k(-1)$

33. $k(x) = -2 \cdot 4^{2x-2};\ k(2)$

34. $p(t) = -2 \cdot 4^{2t+1} + 1;\ p(-2)$

For each given function, evaluate the function at the value or expression and simplify.

35. $h(x) = x^3 + 2;\ h(-4x)$

36. $h(x) = 3x + 2;\ h(-1 + x)$

37. $h(t) = 2| - 3t - 1| + 2;\ h(t^2)$

38. $g(x) = x + 1;\ g(3x)$

39. $g(x) = 5^x;\ g(-3 - x)$

40. $h(n) = 4n + 2;\ h(n + 2)$

41. $h(a) = -3 \cdot 2^{a+3};\ h\left(\dfrac{a}{4}\right)$

42. $h(x) = x^2 + 1;\ h\left(\dfrac{x}{4}\right)$

43. $h(t) = t^2 + t;\ h(t^2)$

44. $h(n) = 5^{n-1} + 1;\ h\left(\dfrac{n}{2}\right)$

5.2 Linear functions

Previously, we discussed graphing linear equations. Putting it all together with functions, we now discuss ***linear functions***. We treat linear functions in the same manner as linear equations, except for the condition that linear functions have only one output for every input.

Definition

A ***linear function*** is a function of the form

$$f(x) = mx + b$$

The graph of a linear function is a *line* and the coefficients m represents the slope of the line and b represents the y-intercept.

We can rewrite the slope formula using function notation as

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

where $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are ordered-pairs on the line.

Example 5.12

If $f(x)$ is a linear function and given $f(7) = 9$ and $f(12) = -2$, determine the linear function.

Solution.

The first thing we want to do is rewrite the function values as ordered-pairs on the line.

$$\begin{aligned} f(7) &= 9 && \text{Rewrite as an ordered-pair} \\ (7, 9)& && \text{This is } (x_1, f(x_1)) \end{aligned}$$

$$\begin{aligned} f(12) &= -2 && \text{Rewrite as an ordered-pair} \\ (12, -2)& && \text{This is } (x_2, f(x_2)) \end{aligned}$$

Next, we see that we have two points in which we need to find the equation of the line. Hence, we use the same techniques as usual, but now in function notation. Let's find the slope using $(7, 9)$ and $(12, -2)$ as the ordered-pairs.

$$\begin{aligned} m &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} && \text{Plug-n-chug the ordered-pairs} \\ m &= \frac{-2 - 9}{12 - 7} && \text{Simplify} \\ m &= \frac{-11}{5} && \text{Slope of the line, } m \end{aligned}$$

Using the point-slope formula, let's plug-n-chug one of the points, $(7, 9)$, and the slope $m = -\dfrac{11}{5}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope formula} \\ y - 9 &= -\frac{11}{5}(x - 7) && \text{Simplify} \\ y - 9 &= -\frac{11}{5}x + \frac{77}{5} && \text{Isolate } y \\ y &= -\frac{11}{5}x + \frac{77}{5} + 9 && \text{Simplify} \\ y &= -\frac{11}{5}x + \frac{122}{5} && \text{Equation of a line} \end{aligned}$$

Notice, we have y isolated on the left. However, we need to rewrite the equation in function notation, so we need to replace the y with $f(x)$:

$$f(x) = -\frac{11}{5}x + \frac{122}{5}$$

Note

In example 5.12, we used the *point-slope formula* to find the equation of the linear function, and then rewrote it, in the final step, using $f(x)$. To use the point-slope formula in function notation, we can use the formula

$$f(x) = m(x - x_1) + y_1$$

where m is the slope, and (x_1, y_1) is a point on the line.

Example 5.13

Let's redo example 5.12 using the function notation of the point-slope formula to find the equation of the line.

Solution.

Using the formula, let's plug-n-chug one of the points, $(7, 9)$, and the slope $m = -\dfrac{11}{5}$.

$$\begin{aligned} f(x) &= m(x - x_1) + y_1 && \text{Point-slope formula in function notation} \\ f(x) &= -\frac{11}{5}(x - 7) + 9 && \text{Simplify} \\ f(x) &= -\frac{11}{5}x + \frac{77}{5} + 9 && \\ f(x) &= -\frac{11}{5}x + \frac{122}{5} && \text{Equation of a line in function notation} \end{aligned}$$

So, we can see the advantage of using the point-slope formula in function notation when trying to obtain a linear function. We easily plug-n-chug the slope and a point, then simplify to obtain the linear function. Students are encouraged to use this formula when appropriate.

5.2.1 Linear functions as applications

We use functions mostly for applications to the real world, usually called *linear modeling*. The slope is no longer thought of as a formula, or *rise over run*, but as an ***average rate of change***. Furthermore, the

y-intercept, b, is thought of as an initial, fixed, or start-up value.

Definition

The ***average rate of change*** for linear functions is represented by the formula

$$m = \frac{\text{change in outputs}}{\text{change in inputs}}$$

and the units are interpreted as [*output units*] *per* [*input units*], e.g., *miles* per *hour*, where miles are the output units and hour is the input unit.

Example 5.14

The cost $C(x)$, where x is the number of miles driven, of renting a car for a day is \$21 plus \$1.05 per mile.

a) What is the slope of the linear function and its units?

Solution.

The slope is the average rate of change where the units for the average rate of change is [*output units*] *per* [*input units*]. From the above given parameters, the slope is 1.05. Its units are dollars per mile.

b) What is the y-intercept and its units?

Solution.

The y-intercept is the initial/fixed/start-up value. In this case, whether the car is driven or not, the daily cost is \$21. Hence, the y-intercept is 20. Its units are dollars.

c) What is the linear function, $C(x)$?

Solution.

The linear function is given as $f(x) = mx + b$, but in this case, we have $C(x) = mx + b$. Since the slope is 1.05 and the y-intercept is 20, then $C(x) = 1.05x + 20$ and its units is dollars.

5.2.2 Graphing linear functions

Now that we've seen and interpreted graphs of linear equations, let's take a look at graphing linear functions. We can use the techniques from a previous chapter: plotting points and then drawing a line through the points or use the y-intercept and slope. We will demonstrate both.

Example 5.15

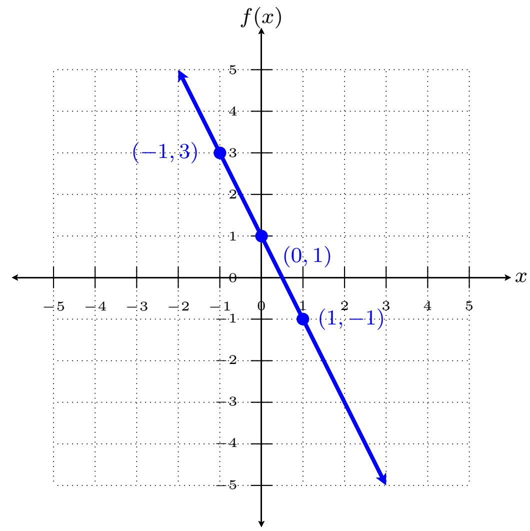
Graph $f(x)$ by point-plotting: $f(x) = -2x + 1$

Solution.

Usually, we pick three x-coordinates, and find corresponding y-values. Each x-value being positive, negative, and zero. This is common practice, but not required.

x	$f(x) = -2x + 1$	$(x, f(x))$
-1	$f(-1) = -2(-1) + 1 = 2 + 1 = 3$	$(-1, 3)$
0	$f(0) = -2(0) + 1 = 0 + 1 = 1$	$(0, 1)$
1	$f(1) = -2(1) + 1 = -2 + 1 = -1$	$(1, -1)$

Plot the three ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Draw the line to fill the grid and put arrows at the ends. It is recommended to purchase a small 6-inch ruler to make nice straight lines.

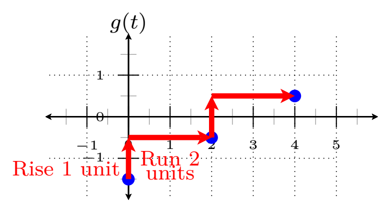


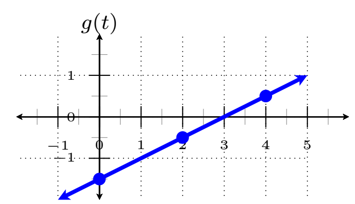
Example 5.16

Graph $g(t) = \frac{1}{2}t - \frac{3}{2}$ by using the slope and y-intercept.

Solution.

The y-intercept, or b, is where the graph crosses the y-axis. The y-intercept is $-\frac{3}{2}$ and the line will cross the y-axis at $\left(0, -\frac{3}{2}\right)$. The slope is $\frac{1}{2}$, and, using $\frac{rise}{run}$, we need to rise upward 1 unit and run to the right 2 units to reach the next point. We continue the pattern to obtain a third point. Now we can connect the dots and create a well-defined line. Be sure to draw it to fill the grid.





Note

Notice when graphing linear functions, it is similar when graphing linear equations. The only difference is the y-axis units changes to its function notation, i.e., in example 5.15, instead of labeling y on the vertical axis, we labeled it $f(x)$.

5.2.3 Linear Functions Homework

1. If $f(x)$ is a linear function and given $f(3) = 2$ and $f(13) = 4$, determine the linear function.

2. If $f(x)$ is a linear function and given $f(4) = -9$ and $f(12) = 3$, determine the linear function.

3. The cost $C(x)$, where x is the number of miles driven, of renting a car for a day is \$50 plus \$1.85 per mile.

- **a)** What is the slope of the linear function and its units?
- **b)** What is the y-intercept and its units?
- **c)** What is the linear function, $C(x)$?

4. The cost $C(x)$, where x is the number of miles driven, of renting a car for a day is \$25 plus \$0.65 per mile.

- **a)** What is the slope of the linear function and its units?
- **b)** What is the y-intercept and its units?
- **c)** What is the linear function, $C(x)$?

Graph each linear function.

5. $f(x) = 2x - 1$

6. $f(x) = 3 - x$

7. $g(x) = \dfrac{1}{3}x - 3$

8. $h(t) = \dfrac{1}{5}t + 1$

9. $p(n) = \dfrac{2}{3}n + \dfrac{1}{3}$

10. $f(t) = \dfrac{1-t}{2}$

11. $k(x) = -2x$

12. $r(t) = 3$

13. $a(n) = 0$

Extending the Concepts: Complete the exercises.

14. Jeff can walk comfortably at 3 miles per hour. Find a linear function d that represents the total distance Jeff can walk in t hours, assuming he doesn't take any breaks.

15. Carl can stuff 6 envelopes per *minute*. Find a linear function E that represents the total number of envelopes Carl can stuff after t *hours*, assuming he doesn't take any breaks.

16. A landscaping company charges \$45 per cubic yard of mulch plus a delivery charge of \$20. Find a linear function which computes the total cost C (in dollars) to deliver x cubic yards of mulch.

17. A plumber charges \$50 for a service call plus \$80 per hour. If she spends no longer than 8 hours a day at any one site, find a linear function that represents her total daily charges C (in dollars) as a function of time t (in hours) spent at any one given location.

18. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find a linear function that represents her total weekly pay, W (in dollars) in terms of x.

19. An on-demand publisher charges \$22.50 to print a 600-page book and \$15.50 to print a 400-page book. Find a linear function which models the cost of a book C as a function of the number of pages p. Interpret the slope of the linear function and find and interpret $C(0)$.

5.3 Algebra of functions

In the previous section, we used the newly defined function notation to make sense of expressions such as $f(x+1)$ or $f(2a)$ for a given function f. It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers, i.e., addition and subtraction.

5.3.1 Add and subtract functions

Definition

If x in the domains of functions f and g, where $(f+g)(x)$ and $(f-g)(x)$ is defined for all x, then

- the sum $f+g$ is given by $(f+g)(x) = f(x) + g(x)$
- the difference $f-g$ is given by $(f-g)(x) = f(x) - g(x)$

Note

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on.

Warning!

Note, that while the formula $(f+g)(x) = f(x) + g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left-hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function $f+g$, the sum of the real number outputs from f and g.

Example 5.17

Let $f(x) = 2x^2 + x - 3$ and $g(x) = -x^2 - 2x + 1$. Find

a) $(f+g)(2)$ **b)** $(f-g)\left(-\frac{3}{2}\right)$ **c)** $(f+g)(x)$ **d)** $(f-g)(x)$

a) $(f+g)(2)$

Solution.

We apply the definition of addition of functions to $(f+g)(2)$, then evaluate.

$$\begin{aligned} (f+g)(2) &= f(2) + g(2) && \text{Apply the defintiion} \\ (f+g)(2) &= \underbrace{\left(2(2)^2 + (2) - 3\right)}_{f(2)} + \underbrace{\left(-(2)^2 - 2(2) + 1\right)}_{g(2)} && \text{Evaluate} \\ (f+g)(2) &= 0 && \text{The sum of the outputs } f(2)+g(2) \end{aligned}$$

b) $(f-g)\left(-\frac{3}{2}\right)$

Solution.

We apply the definition of subtraction of functions to $(f-g)\left(-\frac{3}{2}\right)$, then evaluate.

$$\begin{aligned}
(f-g)\left(-\frac{3}{2}\right) &= f\left(-\frac{3}{2}\right) - g\left(-\frac{3}{2}\right) && \text{Apply the defintiion}\\
(f-g)\left(-\frac{3}{2}\right) &= \underbrace{\left(2\left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right) - 3\right)}_{f(-\frac{3}{2})} - \underbrace{\left(-\left(-\frac{3}{2}\right)^2 - 2\left(-\frac{3}{2}\right) + 1\right)}_{g(-\frac{3}{2})} && \text{Evaluate}\\
(f-g)\left(-\frac{3}{2}\right) &= -\frac{7}{4}
\end{aligned}$$

Thus, the difference of the outputs $f\left(-\frac{3}{2}\right) - g\left(-\frac{3}{2}\right)$.

c) $(f+g)(x)$

Solution.

We apply the definition of addition of functions to $(f+g)(x)$, then simplify by combining like terms.

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) && \text{Apply the defintiion}\\
(f+g)(x) &= \underbrace{\left(2x^2 + x - 3\right)}_{f(x)} + \underbrace{\left(-x^2 - 2x + 1\right)}_{g(x)} && \text{Simplify by combining like terms}\\
(f+g)(x) &= 2x^2 + x - 3 - x^2 - 2x + 1\\
(f+g)(x) &= x^2 - x - 2 && \text{The sum of } \textit{all} \text{ outputs } f(x)+g(x)
\end{aligned}$$

d) $(f-g)(x)$

Solution.

We apply the definition of subtraction of functions to $(f-g)(x)$, then simplify by combining like terms.

$$\begin{aligned}
(f-g)(x) &= f(x) - g(x) && \text{Apply the defintiion}\\
(f-g)(x) &= \underbrace{\left(2x^2 + x - 3\right)}_{f(x)} - \underbrace{\left(-x^2 - 2x + 1\right)}_{g(x)} && \text{Distribute the negative}\\
(f-g)(x) &= 2x^2 + x - 3 + x^2 + 2x - 1 && \text{Simplify by combining like terms}\\
(f+g)(x) &= 3x^2 + 3x - 4 && \text{The difference of } \textit{all} \text{ outputs } f(x)-g(x)
\end{aligned}$$

Note

In these last two examples, essentially we are combining a function with another function. The interesting part about these examples is when we combine a function with another function, we are creating an entirely *new* function. How awesome, right?

5.3.2 Composition of functions

We can think back to Examples 5.10 and 5.11, where we wanted to evaluate $g(3x)$ and $p(t+1)$. After we evaluated these functions, we obtained entirely new functions. In general, when we substitute a function into another function's input, we call this a composition of two functions. We can compose more than two functions, but, in this section, we demonstrate composition only with two functions.

Definition

If x is in the domains of functions f and g, then f ***composed*** of g is given by

$$(f \circ g)(x) = f(g(x)),$$

i.e, we substitute every x in f with the function $g(x)$.

$(f \circ g)(x)$ implies that x is in the domain of $g(x)$ and $g(x)$ is in the domain of $f(x)$.

World Note

The term *function* came from Gottfried Wihelm Leibniz, a German mathematician from the late 17^{th} century.

Let's look at a graphical representation of the above definition.

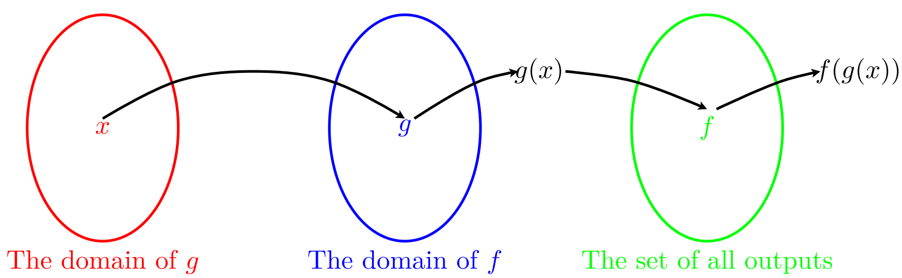


Figure 5.1: In the figure, we see the composition of g and x followed by f. Thus, $(f \circ g)(x)$ is composing x into g first, then composing $g(x)$ into f, getting an entirely new function.

Example 5.18

If $a(x) = x^2 - 2x + 1$ and $b(x) = x - 5$, find $(a \circ b)(3)$.

Solution.

Let's rewrite $(a \circ b)(3)$ using the definition, then evaluate.

$$\begin{aligned}
(a \circ b)(3) &= a(b(3)) && \text{Apply the definition} \\
(a \circ b)(3) &= a(\underbrace{3-5}_{b(3)}) && \text{Simplify } 3-5 \\
(a \circ b)(3) &= a(-2) && \text{Evaluate } a(-2) \\
(a \circ b)(3) &= (-2)^2 - 2(-2) + 1 && \text{Simplify} \\
(a \circ b)(3) &= 9 && \text{The value of the composition}
\end{aligned}$$

Recall, when we evaluate functions at particular values, we are really obtaining ordered-pairs on the graph of the function. Since we are composing two functions in this case, then the ordered-pair of the *new* function is $(3, 9)$.

Example 5.19

Let $f(x) = x^2 - x$ and $g(x) = x + 3$.

a) Find $(f \circ g)(x)$.

Solution.

We start by rewriting $(f \circ g)(x)$ using the definition, then simplify. Note, after we simplify, we obtain an entirely new function.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Replace } g(x) \text{ with } x+3 \\
(f \circ g)(x) &= f(\underbrace{x+3}_{g(x)}) && \text{Replace the variables in } f \text{ with } (x+3) \\
(f \circ g)(x) &= (x+3)^2 - (x+3) && \text{Simplify} \\
(f \circ g)(x) &= (x^2 + 6x + 9) - (x+3) && \text{Combine like terms} \\
(f \circ g)(x) &= x^2 + 5x + 6 && \text{The composition of } f \text{ and } g
\end{aligned}$$

As we assumed, composing f and g resulted in entirely *new* function.

b) Find $(g \circ f)(x)$.

Solution.

Notice, $(g \circ f)(x)$, where we are composing g and f. Just like the chocolate covered strawberries with whipped cream, *order matters*. We will have to put f in g, unlike the previous example.

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) && \text{Replace } f(x) \text{ with } x^2 - x \\
(g \circ f)(x) &= g\underbrace{(x^2 - x)}_{f(x)} && \text{Replace the variables in } g \text{ with } (x^2 - x) \\
(g \circ f)(x) &= \left(x^2 - x\right) + 3 && \text{Simplify} \\
(g \circ f)(x) &= x^2 - x + 3 && \text{The composition of } g \text{ and } f
\end{aligned}$$

As we assumed, composing g and f, where we dipped f into g resulted in entirely *different* function than the previous example.

c) Find $(f \circ f)(x)$.

Solution.

Notice, $(f \circ f)(x)$, where we are composing f with itself. We will substitute the function f into itself for every x.

$$\begin{aligned}
(f \circ f)(x) &= f(f(x)) && \text{Replace } f(x) \text{ with } x^2 - x \\
(f \circ f)(x) &= f\underbrace{(x^2 - x)}_{f(x)} && \text{Replace the variables in } f \text{ with } (x^2 - x) \\
(f \circ f)(x) &= \left(x^2 - x\right)^2 - \left(x^2 - x\right) && \text{Simplify} \\
(f \circ f)(x) &= x^4 - 2x^3 + x^2 - x^2 + x && \text{Combine like terms} \\
(f \circ f)(x) &= x^4 - 2x^3 + x && f \text{ composed with itself}
\end{aligned}$$

Notice, even when we compose a function with itself, it still results in an entirely *new* function.

✎ **Note**

With composition of functions, essentially, we are composing a function with another function. The interesting part about these examples is when we compose a function with another function, we are creating an entirely *new* function.

5.3.3 Algebra of Functions Homework

Given the functions, perform the indicated operations and simplify.

1. $g(a) = a^3 + 5a^2$ and $f(a) = 2a + 4$; find $g(3) + f(3)$

2. $g(a) = 3a + 3$ and $f(a) = 2a - 2$; find $(g + f)(9)$

3. $g(x) = x + 3$ and $f(x) = -x + 4$; find $(g - f)(3)$

4. $g(x) = x^2 + 2$ and $f(x) = 2x + 5$; find $(g - f)(0)$

5. $g(t) = t - 3$ and $h(t) = -3t^3 + 6t$; find $g(1) + h(1)$

6. $h(t) = t + 5$ and $g(t) = 3t - 5$; find $(h - g)(5)$

7. $h(n) = 2n - 1$ and $g(n) = 3n - 5$; find $h(0) + g(0)$

8. $f(a) = -2a - 4$ and $g(a) = a^2 + 3$; find $(f + g)(a)$

9. $g(x) = -x^3 - 2$ and $h(x) = 4x$; find $(g - h)(x)$

10. $f(x) = -3x + 2$ and $g(x) = x^2 + 5x$; find $(f - g)(x)$

11. $g(x) = 4x + 5$ and $h(x) = x^2 + 5x$; find $(g + h)(x)$

12. $f(x) = -3x^2 + 3x$ and $g(x) = 2x + 5$; find $(f + g)(-4)$

13. $g(x) = 4x + 3$ and $h(x) = x^3 - 2x^2$; find $(g - h)(-1)$

14. $g(x) = -4x + 1$ and $h(x) = -2x - 1$; find $g(5) + h(5)$

15. $f(n) = n - 5$ and $g(n) = 4n + 2$; find $(f + g)(-8)$

16. $g(a) = 3a - 2$ and $h(a) = 4a - 2$; find $(g + h)(-10)$

17. $g(x) = x^2 - 2$ and $h(x) = 2x + 5$; find $g(-6) + h(-6)$

18. $g(n) = n^2 - 3$ and $h(n) = 2n - 3$; find $(g - h)(n)$

19. $g(x) = 2x - 3$ and $h(x) = x^3 - 2x^2 + 2x$; find $(g - h)(x)$

20. $f(x) = 2x$ and $g(x) = -3x - 1$; find $(f + g)(-4 - x)$

21. $f(t) = t^2 + 4t$ and $g(t) = 4t + 2$; find $f(t^2) + g(t^2)$

22. $f(n) = -3n^2 + 1$ and $g(n) = 2n + 1$; find $(f - g)\left(\frac{n}{3}\right)$

23. $f(x) = -4x + 1$ and $g(x) = 4x + 3$; find $(f \circ g)(9)$

24. $h(a) = 3a + 3$ and $g(a) = a + 1$; find $(h \circ g)(5)$

25. $g(x) = x + 4$ and $h(x) = x^2 - 1$; find $(g \circ h)(10)$

26. $f(x) = 4x - 4$ and $g(x) = 3x^2 - 5$; find $(f + g)(x)$

27. $f(x) = 2x + 4$ and $g(x) = 4x - 5$; find $f(x) - g(x)$

28. $g(t) = t^3 + 3t^2$ and $h(t) = 3t - 5$; find $g(t) - h(t)$

29. $f(n) = 3n + 4$ and $g(n) = n^3 - 5n$; find $f\left(\frac{n}{2}\right) - g\left(\frac{n}{2}\right)$

30. $g(x) = x - 1$; find $(g \circ g)(7)$

31. $g(t) = t + 3$ and $h(t) = 2t - 5$; find $(g \circ h)(3)$

32. $f(a) = 2a - 4$ and $g(a) = a^2 + 2a$; find $(f \circ g)(-4)$

33. $f(n) = -4n + 2$ and $g(n) = n + 4$; find $(f \circ g)(9)$

34. $g(x) = 2x - 4$ and $h(x) = 2x^3 + 4x^2$; find $(g \circ h)(3)$

35. $g(x) = x^2 - 5x$ and $h(x) = 4x + 4$; find $(g \circ h)(x)$

36. $f(a) = -2a + 2$ and $g(a) = 4a$; find $(f \circ g)(a)$

37. $g(x) = 4x + 4$ and $f(x) = x^3 - 1$; find $(g \circ f)(x)$

38. $g(x) = -x + 5$ and $f(x) = 2x - 3$; find $(g \circ f)(x)$

39. $f(t) = 4t + 3$ and $g(t) = -4t - 2$; find $(f \circ g)(t)$

40. $g(x) = 3x + 4$ and $h(x) = x^3 + 3x$; find $(g \circ h)(3)$

41. $g(a) = a^2 + 3$; find $(g \circ g)(-3)$

42. $g(a) = 2a + 4$ and $h(a) = -4a + 5$; find $(g \circ h)(a)$

43. $g(t) = -t - 4$; find $(g \circ g)(t)$

44. $f(n) = -2n^2 - 4n$ and $g(n) = n + 2$; find $(f \circ g)(n)$

45. $g(t) = t^3 - t$ and $f(t) = 3t - 4$; find $(g \circ f)(t)$

46. $f(x) = 3x - 4$ and $g(x) = x^3 + 2x^2$; find $(f \circ g)(x)$

5.4 Library of functions

Knowing graphs of common functions assist the student with visualizing the function and connecting the relationship between the equation and its graph.

Library of functions

The ***library of functions*** is a set of functions that distinguishes the relationship between the functions and their graphs which includes the domain for each function.

The library of functions grows as we become more familiar with different types of functions. As we take more higher-level mathematics, the library grows to be very large, but for this section, we begin with a library that contains six important basic functions: line, parabola, cubic, absolute value, rational, square root.

✎ **Note**

The graphs of the functions in the library of functions are the general graphs of the functions, not particular graphs of functions. Hence, we can use point-plotting, technology, or transformations to graph particular functions, but we tend to memorize the general form as it is helpful in higher-level mathematics to recall the library of functions quickly.

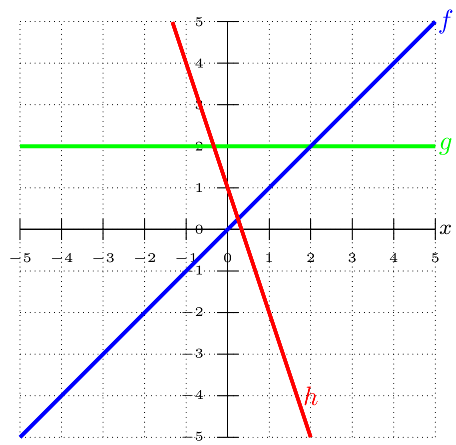
Example 5.20

Graph $f(x) = x$, $g(x) = 2$, and $h(x) = -3x + 1$ and determine their domain.

Solution.

Notice, all three functions are linear functions. We can plot them easily on the same grid.

We can see that all graphs are lines and since there are no restrictions to any of the lines, the domain is all real numbers or $(-\infty, \infty)$. Since f is a line through the origin (y-intercept is zero), and every x coordinate is the same as its corresponding y coordinate, e.g., $(0, 0)$, $(1, 1)$, etc., then we call f the ***identity function***.



Since g is a horizontal line, the y coordinates never change, and there isn't a change in slope, i.e., the slope is zero, then we call g the ***constant function***.

The function h is an equation of a line with a non-zero slope and non-zero y-intercept, and we call h a ***linear function***.

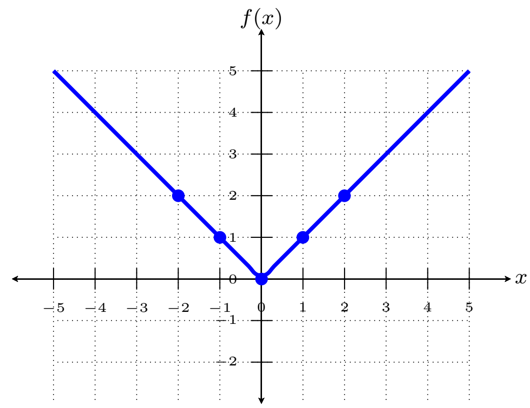
Example 5.21

Graph $f(x) = |x|$ and determine the domain.

Solution.

Let's pick five x-coordinates, and find corresponding y-values. Each x-value being positive or negative, and zero. This is common practice, but not required.

x	$f(x) = \lvert x\rvert$	$(x, f(x))$
-2	$f(-2) = 2$	$(-2, 2)$
-1	$f(-1) = 1$	$(-1, 1)$
0	$f(0) = 0$	$(0, 0)$
1	$f(1) = 1$	$(1, 1)$
2	$f(2) = 2$	$(2, 2)$



Plot the five ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. This graph looks like two lines of opposite slopes that meet at the origin. Hence, it's graph is two lines that meet at the origin, but stop where it meets to make a v-shape called an ***absolute value function*** Since we see there are no restrictions to the graph, the domain is all real numbers or $(-\infty, \infty)$.

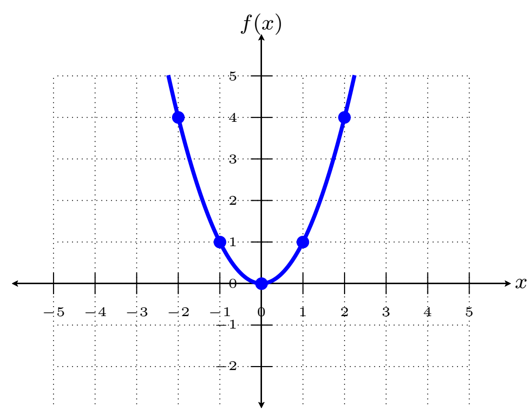
Example 5.22

Graph $f(x) = x^2$ and determine the domain.

Solution.

Let's pick five x-coordinates, and find corresponding y-values. Each x-value being positive, negative, and zero. This is common practice, but not required.

x	$f(x) = x^2$	$(x, f(x))$
-2	$f(-2) = 4$	$(-2, 4)$
-1	$f(-1) = 1$	$(-1, 1)$
0	$f(0) = 0$	$(0, 0)$
1	$f(1) = 1$	$(1, 1)$
2	$f(2) = 4$	$(2, 4)$



Plot the five ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. This graph is called a *parabola* and since this function is quite common for the x^2-form, we call it a ***quadratic (square) function***. Since we see there are no restrictions to the graph, the domain is all real numbers or $(-\infty, \infty)$.

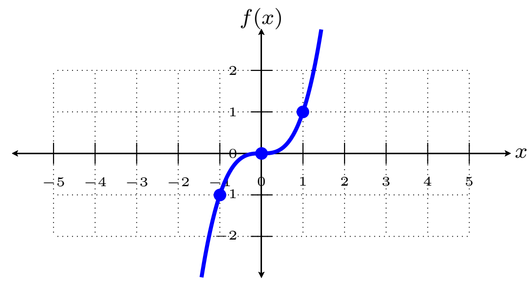
Example 5.23

Graph $f(x) = x^3$ and determine the domain.

Solution.

Let's pick three x-coordinates, and find corresponding y-values. Each x-value being positive, negative, and zero. This is common practice, but not required.

x	$f(x) = x^3$	$(x, f(x))$
-1	$f(-1) = -1$	$(-1, -1)$
0	$f(0) = 0$	$(0, 0)$
1	$f(1) = 1$	$(1, 1)$



Plot the ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Since this function is quite common for the x^3-form, we call it a ***cube (cubic) function***. Since we see there are no restrictions to the graph, the domain is all real numbers or $(-\infty, \infty)$.

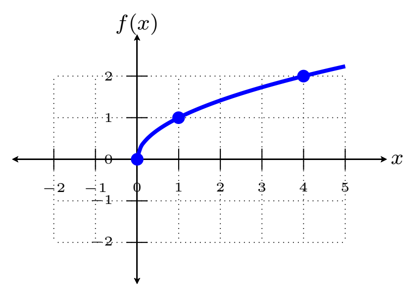
Example 5.24

Graph $f(x) = \sqrt{x}$ and determine the domain.

Solution.

Let's pick three x-coordinates, and find corresponding y-values.

x	$f(x) = \sqrt{x}$	$(x, f(x))$
0	$f(0) = 0$	$(0, 0)$
1	$f(1) = 1$	$(1, 1)$
4	$f(4) = 2$	$(4, 2)$



Plot the ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Since this function is quite common for the $\sqrt{x}$-form, we call it a ***square root function***. Since we see there is one restriction to the graph, where the x values start at the origin and no part of the graph is on the left side of the origin, the domain of this function is $\{x|x \geq 0\}$ or $[0, \infty)$.

Example 5.25

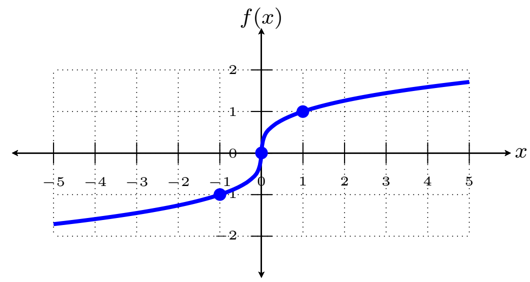
Graph $f(x) = \sqrt[3]{x}$ and determine the domain.

Solution.

Let's pick three x-coordinates, and find corresponding y-values.

x	$f(x) = \sqrt[3]{x}$	$(x, f(x))$
-1	$f(-1) = -1$	$(-1, -1)$
0	$f(0) = 0$	$(0, 0)$
1	$f(1) = 1$	$(1, 1)$

Plot the ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. This function looks like the cube function, but flipped and $90°$ to the right! We call this function a ***cube root function***. because of the root on the radical. Since we see there are no restrictions to the graph, the domain is all real numbers or $(-\infty, \infty)$.



Example 5.26

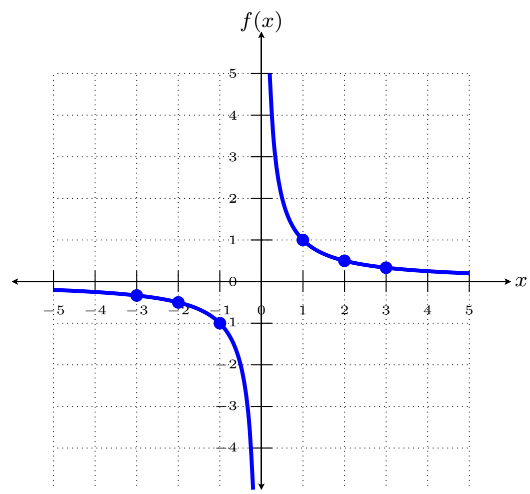
Graph $f(x) = \dfrac{1}{x}$ and determine the domain.

Solution.

Let's pick x-coordinates, and find corresponding y-values.

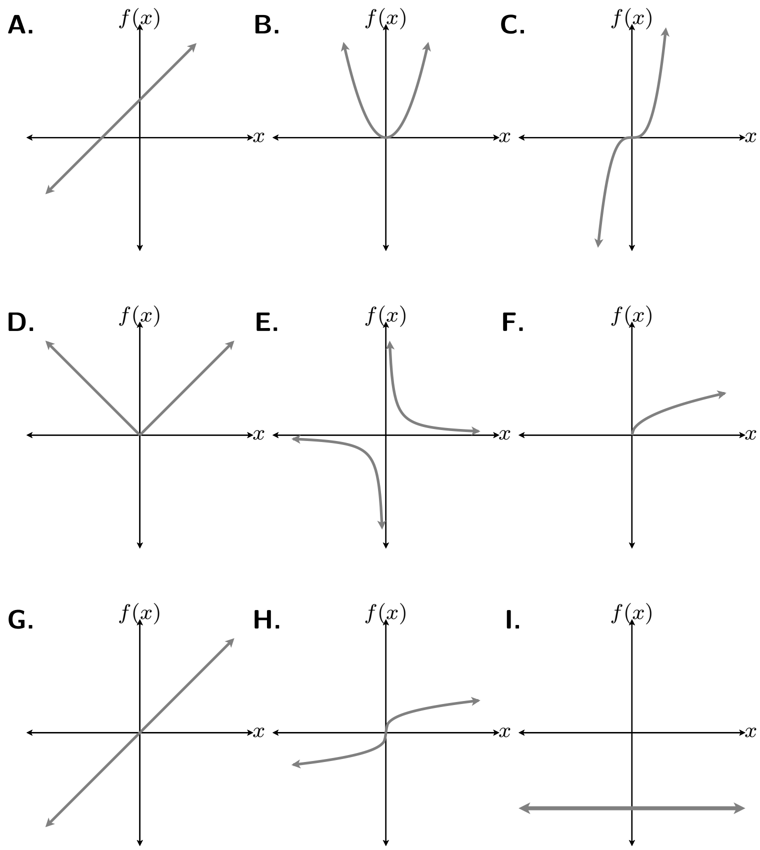
x	$f(x) = \frac{1}{x}$	$(x, f(x))$
-3	$f(-3) = -\frac{1}{3}$	$(-3, -\frac{1}{3})$
-2	$f(-2) = -\frac{1}{2}$	$(-2, -\frac{1}{2})$
-1	$f(-1) = -1$	$(-1, -1)$
0	$f(0) =$ undefined	no point
1	$f(1) = 1$	$(1, 1)$
2	$f(2) = \frac{1}{2}$	$(2, \frac{1}{2})$
3	$f(3) = \frac{1}{3}$	$(3, \frac{1}{3})$

Plot the ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. This graph looks most different than the other functions and it is because it is a fraction with a variable in the denominator. Recall, fractions cannot have zero in their denominators because that is when fractions are undefined. We will learn more about these functions that are called *rational functions*. For now, we call the graph of this function a ***reciprocal function***. Since we see that this function cannot have zero in the denominator, and, from the table, we see when $x = 0$, the function is undefined, then the domain is all real numbers except for $x = 0$: $\{x | x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$.



5.4.1 Library of Functions Homework

Given below are the graphs of functions.



1. Match each graph with the *name of its function.*

Names of the Functions

a) Reciprocal Function
b) Absolute Value Function
c) Cube Root Function
d) Cube Function
e) Constant Function
f) Identity Function
g) Square Root Function
h) Square Function
i) Linear Function

2. Match each graph with the *formula of the function.*

Formulas of Functions

a) $f(x) = x$
b) $f(x) = -4$
c) $f(x) = |x|$
d) $f(x) = x^2$
e) $f(x) = \dfrac{1}{x}$
f) $f(x) = x^3$
g) $f(x) = mx + b$
h) $f(x) = \sqrt[3]{x}$
i) $f(x) = \sqrt{x}$

3. What are the domains for each of the functions?

5.5 Function: Answers to the Homework Exercises

Introductions to Functions

1. All are functions, except **B.** and **G.** because they fail the vertical line test

3. domain: $(-\infty, 0) \cup (0, \infty)$; t is the independent variable, $s(t)$ is the dependent variable

5. domain: $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$; x is the independent variable, $f(x)$ is the dependent variable

7. domain: $(-\infty, \frac{5}{4}]$; x is the independent variable, $f(x)$ is the dependent variable

9. domain: $[16, \infty)$; x is the independent variable, $f(x)$ is the dependent variable

11. domain: $(-\infty, \infty)$; t is the independent variable and its units are seconds, $h(t)$ is the dependent variable and its units are meters

13. domain: $(-\infty, \infty)$; x is the independent variable and its units are the number of hearing aids, $R(x)$ is the dependent variable and its units are dollars.

15. $g(0) = -4$; $(0, -4)$

17. $f(-6) = -7$; $(-6, -7)$

19. $f(10) = 13$; $(10, 13)$

21. $w(-2) = \frac{9}{4}$; $\left(-2, \frac{9}{4}\right)$

23. $p(4) = -60$; $(4, -60)$

25. $g(2) = -\frac{3}{25}$; $(2, -\frac{3}{25})$

27. $f(10) = 7$; (10.7)

29. $w(-5) = 5$; $(-5, 5)$

31. $p(5) = -4$; $(5, -4)$

33. $k(2) = -32$; $(2, -32)$

35. $-64x^3 + 2$

37. $2|-3t^2 - 1| + 2$

39. 5^{-3-x}

41. $3 \cdot 2^{a/4+3}$

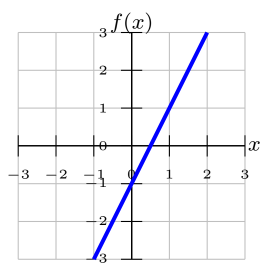
43. $t^4 + t^2$

Linear Functions

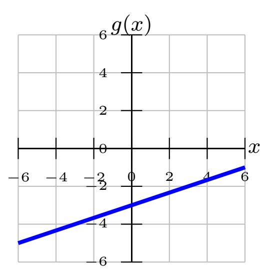
1. $f(x) = \frac{1}{5}x + \frac{7}{5}$

3. a) 1.85; dollars per mile; b) 50; dollars; c) $C(x) = 1.85x + 50$

5.



7.



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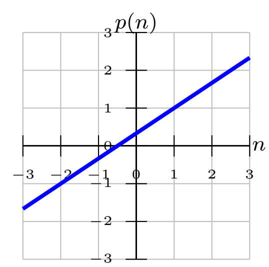

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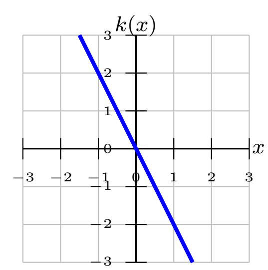
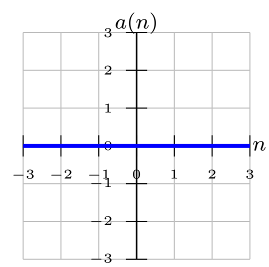


13.



15. $E(t) = 360t$

17. $C(t) = 80t + 50$, where $0 < t \leq 8$

19. $C(p) = 0.035p + 1.50$; it costs \$0.035 per page to print; the fixed costs are \$1.50 per book whether pages are printed or not

Algebra of Functions

1. 82

3. 5

5. 1

7. $\frac{1}{5}$

9. $-x^3 - 4x - 2$

11. $4x^3 + 25x^2 + 25x$

13. 2

15. -43

17. 27

19. $-x^3 + 2x^2 - 3$

21. $t^4 + 8t^2 + 2$

23. -155

25. 103

27. $-2x + 9$

29. $\frac{32+23n-n^3}{8}$

31. 4

33. -50

35. $16x^2 + 12x - 4$

37. $4x^3$

39. $-16t - 5$

41. 147

43. t

45. $27t^3 - 108t^2 + 141t - 60$

Library of Functions

1. A $\leftrightarrow$ i; B $\leftrightarrow$ h; C $\leftrightarrow$ d; D $\leftrightarrow$ b; E $\leftrightarrow$ a; F $\leftrightarrow$ g; G $\leftrightarrow$ f; H $\leftrightarrow$ c; I $\leftrightarrow$ e

2. A $\leftrightarrow$ g; B $\leftrightarrow$ d; C $\leftrightarrow$ f; D $\leftrightarrow$ c; E $\leftrightarrow$ e; F $\leftrightarrow$ i; G $\leftrightarrow$ a; H $\leftrightarrow$ h; I $\leftrightarrow$ b

3. A, B, C, D, G, H, I $\to (-\infty, \infty)$, E $\to (-\infty, 0) \cup (0, \infty)$, F $\to (0, \infty)$

Chapter 6

Exponents and Polynomials

Chapter Objectives

By the end of this chapter, the student should be able to

- Simplify exponential expressions with positive and/or negative exponents
- Multiply or divide expressions in scientific notation
- Evaluate polynomials for specific values
- Apply arithmetic operations to polynomials
- Apply special-product formulas to multiply polynomials
- Divide a polynomial by a monomial, applying long division, synthetic division

In order to apply more involved properties to polynomials, we first discuss a variety of properties of exponents. Problems with exponents can often be simplified using a few basic exponent properties. Exponents represent repeated multiplication and we can use this fact to discover the properties of exponents.

World Note

The word *exponent* comes from the Latin "expo" meaning *out of*, and "ponere" meaning *to place*. While there is some debate, it seems that the Babylonians living in Iraq were the first to do work with exponents dating back to the 23^{rd} century BC or earlier.

6.1 Exponents rules and properties

Definition

If a is a positive real number and n is any real number, then in

$$a^n,$$

a is called the base and n is called the exponent.

Note

When the directions state simplify, this means

- All exponents are positive
- Each base only occurs once
- There are no parenthesis
- There are no powers written to powers

6.1.1 Product rule of exponents

Let's take a look at an example with multiplication.

Example 6.1

Simplify: $a^3 \cdot a^2$

Solution.

First, let's rewrite this product in expanded form and then combine with one base a.

$$\begin{array}{rl} a^3 \cdot a^2 & \text{Expand} \\ (a \cdot a \cdot a) \cdot (a \cdot a) & \text{Rewrite with one base } a \\ \underbrace{a \cdot a \cdot a \cdot a \cdot a}_{\text{5 times}} & \text{Multiplying } a \text{ five times} \\ a^5 & \text{Simplified expression} \end{array}$$

Let's think about example 6.1. This method of expanding seems to be fine when there are smaller exponents, but what if we were given something like $a^{100} \cdot a^{934}$? Are we going to expand a over a thousand times? No way! We need a more sophisticated way in multiplying expressions with exponents. Hence, taking a look at example 6.1, we can see the result is a^5. Notice we could have obtained this answer without expanding but by simply adding the exponents:

$$a^3 \cdot a^2 = a^{3+2} = a^5$$

This is called the *product rule of exponents*.

Product rule for exponents

Let a be a positive real number and n and m be any real number. Then

$$a^n \cdot a^m = a^{n+m}$$

Note. In order to add exponents, the bases of the factors are required to be the same.

Example 6.2

Simplify: $3^2 \cdot 3^6 \cdot 3$

Solution.

Let's apply the product rule and simplify. Don't forget that 3 has an exponent, it is one: 3^1.

We don't always write it, but we know it's there.

$$\begin{aligned} 3^2 \cdot 3^6 \cdot 3^1 &\quad \text{Same base} \\ 3^{2+6+1} &\quad \text{Add the exponents} \\ 3^9 &\quad \text{Simplified expression} \end{aligned}$$

We can simplify this even more as 19,683 ($3^9 = 19683$).

Example 6.3

Simplify: $\left(2x^3y^5z\right) \cdot \left(5xy^2z^3\right)$

Solution.

$$\begin{aligned} \left(2x^3y^5z\right) \cdot \left(5xy^2z^3\right) &\quad \text{Rewrite without parenthesis} \\ 2x^3y^5z^1 \cdot 5x^1y^2z^3 &\quad \text{Multiply the coefficients and add exponents with same bases} \\ 2 \cdot 5 \cdot x^{3+1} \cdot y^{5+2} \cdot z^{1+3} &\quad \text{Add exponents and multiply the coefficients} \\ 10x^4y^7z^4 &\quad \text{Simplified expression} \end{aligned}$$

6.1.2 Quotient rule of exponents

Example 6.4

Simplify: $\dfrac{a^5}{a^2}$

Solution.

$$\begin{aligned} \frac{a^5}{a^2} &\quad \text{Expand} \\ \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} &\quad \text{Reduce the common factors} \\ \frac{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a}{\cancel{a} \cdot \cancel{a}} &\quad \text{Simplify} \\ a \cdot a \cdot a &\quad \text{Rewrite with one base } a \\ a^3 &\quad \text{Simplified expression} \end{aligned}$$

Let's think about example 6.4. This method of expanding seems to be fine when there are smaller exponents, but what if we were given something like $\frac{a^{199}}{a^{827}}$? Are we going to expand a over a thousand times? No way! We need a more sophisticated way in dividing expressions with exponents. Hence, taking a look at example 6.4, we can see the result is a^3. Notice we could have obtained this answer without expanding but by simply subtracting the exponents:

$$\frac{a^5}{a^2} = a^{5-2} = a^3$$

This is called the *quotient rule of exponents*.

Quotient rule for exponents

Let a be a positive real number and n and m be any real number. Then

$$\frac{a^n}{a^m} = a^{n-m}$$

Note. In order to subtract exponents, the bases of the dividend and divisor are required to be the same. Be sure that the *denominator* exponent is *subtracted from the numerator* exponent.

Example 6.5

Simplify: $\dfrac{7^{13}}{7^5}$

Solution.

$$\begin{array}{rl} \dfrac{7^{13}}{7^5} & \text{Same base} \\ 7^{13-5} & \text{Subtract exponents} \\ 7^8 & \text{Simplified expression} \end{array}$$

We can simplify this even more as 5,764,801 ($7^8 = 5764801$).

Example 6.6

Simplify: $\dfrac{5a^3b^5c^2}{2ab^3c}$

Solution.

$$\begin{array}{rl} \dfrac{5a^3b^5c^2}{2a^1b^3c^1} & \text{Subtract exponents with same bases and simplify coefficients, if possible} \\ \dfrac{5a^{3-1}b^{5-3}c^{2-1}}{2} & \text{Simplify} \\ \dfrac{5a^2b^2c}{2} & \text{Simplified expression} \end{array}$$

We could also write the expression with the fraction as a coefficient: $\dfrac{5}{2}a^2b^2c$. These are equivalent and both correct.

6.1.3 Power rule of exponents

Example 6.7

Simplify: $\left(a^2\right)^3$

Solution.

First, let's rewrite this expression in expanded form and then combine with one base a.

$$\begin{aligned} (a^2)^3 &\quad \text{Expand} \\ a^2 \cdot a^2 \cdot a^2 &\quad \text{Apply the product rule} \\ a^{2+2+2} &\quad \text{Add exponents} \\ a^6 &\quad \text{Simplified expression} \end{aligned}$$

Let's think about example 6.7. This method of expanding seems to be fine when there are smaller exponents, but what if we were given something like $(a^{760})^{34}$? Are we going to expand a over a twenty-thousand times? No way! We need a more sophisticated way in simplifying expressions with exponents raised to exponents. Hence, taking a look at example 6.7, we can see the result is a^6. Notice we could have obtained this answer without expanding but by simply multiplying the exponents:

$$(a^2)^3 = a^{2\cdot 3} = a^6$$

This is called the *power rule of exponents*.

Power rule for exponents

Let a be a positive real number and n and m be any real number. Then

$$(a^n)^m = a^{n\cdot m}$$

Furthermore, we can extend the power rule for when we have more than one factor in the base.

Example 6.8

Simplify: $(ab)^3$

Solution.

We can expand the base, then rewrite with one base of a and b.

$$\begin{aligned} (ab)^3 &\quad \text{Expand} \\ (ab)(ab)(ab) &\quad \text{Let's rewrite this grouping } a\text{'s and } b\text{'s} \\ a \cdot a \cdot a \cdot b \cdot b \cdot b &\quad \text{Rewrite with one base of } a \text{ and } b \\ a^3b^3 &\quad \text{Simplified expression} \end{aligned}$$

Let's think about example 6.8. This method of expanding seems to be fine when there are smaller exponents, but what if we were given something like $(ab)^{2049}$? Are we going to expand a and b over a two-thousand times? No way! We need a more sophisticated way in simplifying expressions with exponents raised to exponents with more than one factor in the base. Hence, taking a look at example 6.8, we can see the result is a^3b^3. Notice we could have obtained this answer without expanding but by simply applying the exponent to each factor in the base:

$$(ab)^3 = a^3 \cdot b^3 = a^3b^3$$

This is called the *power of a product rule* (POP).

Power of a product rule (POP)

Let a and b be a positive real numbers and n be any real number. Then

$$\big(ab\big)^n = a^n \cdot b^m$$

Warning!

It is important to be careful to only use the power of a product rule with multiplication inside parenthesis. This property is not allowed for addition or subtraction, i.e.,

$$(a+b)^m \neq a^m + b^m$$

Example 6.9

Simplify: $\left(\frac{a}{b}\right)^3$

Solution.

Let's expand the fraction and rewrite with one base of a and b.

$\left(\frac{a}{b}\right)^3$	Expand
$\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)$	Multiply fractions
$\frac{a^3}{b^3}$	Simplified expression

Notice, this is similar to the POP rule and we can apply the exponent to each numerator and denominator.

Power of a quotient rule

Let a and b be a positive real numbers and n be any real number. Then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Let's look at an example where we have to combine all these exponent rules.

Example 6.10

Simplify: $(x^3yz^2)^4$

Solution.

$(x^3y^1z^2)^4$	Apply the POP rule
$x^{3\cdot4}y^{1\cdot4}z^{2\cdot4}$	Multiply exponents
$x^{12}y^4z^8$	Simplified expression

Example 6.11

Simplify: $\left(\frac{a^3b}{c^8d^5}\right)^2$

Solution.

$$\left(\frac{a^3b^1}{c^8d^5}\right)^2 \quad \text{Apply the power of a quotient rule}$$

$$\frac{a^{3\cdot2}b^{1\cdot2}}{c^{8\cdot2}d^{5\cdot2}} \quad \text{Multiply exponents}$$

$$\frac{a^6b^2}{c^{16}d^{10}} \quad \text{Simplified expression}$$

Example 6.12

Simplify: $(4x^2y^5)^3$

Solution.

$$\begin{aligned} (4^1x^2y^5)^3 &\quad \text{Apply the POP rule} \\ 4^{3\cdot1}x^{2\cdot3}y^{5\cdot3} &\quad \text{Multiply exponents} \\ 4^3\cdot x^6\cdot y^{15} &\quad \text{Evaluate } 4^3 \\ 64x^6y^{15} &\quad \text{Simplified expression} \end{aligned}$$

Notice that the exponent also applied to the coefficient 4 and we had to evaluate $4^3 = 64$ as part of the expression.

Exponent Rules

Let a and b be positive real numbers and n and m be any real numbers.

Rule 1. $a^n \cdot a^m = a^{n+m}$

Rule 2. $\dfrac{a^n}{a^m} = a^{n-m}$

Rule 3. $(a^n)^m = a^{nm}$

Rule 4. $(ab)^n = a^n \cdot b^n$

Rule 5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

6.1.4 Zero as an exponent

Here we discuss zero as an exponent. This is one of two cases where the exponent isn't positive. The other case is where the exponents are negative, but we will save that for the next section. Let's look an example:

Example 6.13

Simplify: $\dfrac{a^3}{a^3}$

Solution.

If we applied the quotient rule right away, we would get

$$\begin{aligned} \frac{a^3}{a^3} &= a^{3-3} \\ &= a^0 \end{aligned}$$

But what does this mean? What is a^0? Well, let's take a look at this same example with a different approach:

$$\begin{array}{rl} \dfrac{a^3}{a^3} & \text{Expand} \\[2ex] \dfrac{a \cdot a \cdot a}{a \cdot a \cdot a} & \text{Reduce common factors of } a \\[2ex] \dfrac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} & \text{Simplify} \\[2ex] \dfrac{1}{1} & \text{Simplify} \\[2ex] 1 & \text{Simplified expression} \end{array}$$

If $\dfrac{a^3}{a^3} = a^0$ from the first part and $\dfrac{a^3}{a^3} = 1$ from the second part, then this implies $a^0 = 1$.

Zero power rule

Let a be a positive real number. Then $a^0 = 1$, i.e., any positive real number to the power of zero is 1.

Example 6.14

Simplify: $\left(3x^2\right)^0$

Solution.

Since $3x^2$ is raised to the power of zero, then we can apply the zero power rule:

$$\begin{array}{rl} \left(3x^2\right)^0 & \text{Zero power rule} \\ 1 & \text{Simplified expression} \end{array}$$

6.1.5 Negative exponents

Another property we consider is expressions with negative exponents.

Example 6.15

Simplify: $\dfrac{a^3}{a^5}$

Solution.

If we applied the quotient rule right away, we would get

$$\begin{aligned} \frac{a^3}{a^5} &= a^{3-5} \\ &= a^{-2} \end{aligned}$$

But what does this mean? What is a^{-2}? Well, let's take a look at this same example with a different approach:

$$\begin{aligned} &\frac{a^3}{a^5} && \text{Expand} \\ &\frac{a\cdot a\cdot a}{a\cdot a\cdot a\cdot a\cdot a} && \text{Reduce common factors of } a \\ &\frac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot a\cdot a} && \text{Simplify} \\ &\frac{1}{a\cdot a} && \text{Simplify} \\ &\frac{1}{a^2} && \text{Simplified expression} \end{aligned}$$

If $\dfrac{a^3}{a^5} = a^{-2}$ from the first part and $\dfrac{a^3}{a^5} = \dfrac{1}{a^2}$ from the second part, then this implies $a^{-2} = \dfrac{1}{a^2}$.

This example illustrates an important property of exponents. Negative exponents yield *the reciprocal of the base*. Once we take the reciprocal, the exponent is now positive.

Warning!

It is important to note a negative exponent does not imply the expression is negative, only the reciprocal of the base. Hence, negative exponents imply reciprocals.

Also, recall the rules of simplifying:

- All exponents are positive
- Each base only occurs once
- There are no parenthesis
- There are no powers written to powers

This includes rewriting all negative exponents as positive exponents.

Negative exponents rules

Let a and b be positive real numbers and n be any real number.

Rule 1. $a^{-n} = \dfrac{1}{a^n}$ **Rule 2.** $\dfrac{1}{a^{-n}} = a^n$ **Rule 3.** $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

Negative exponents are combined in several different ways. As a general rule, in a fraction, a base with a negative exponent moves to the other side of the fraction bar as the exponent changes sign.

Example 6.16

Simplify: $\dfrac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2}$

Solution.

We can rewrite the expression with positive exponents using the rules of exponents:

$$\frac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2} \qquad \text{Reciprocate the terms with negative exponents}$$

$$\frac{a^3cde^4}{2b^2f^2} \qquad \text{Simplified expression}$$

As we simplified the fraction, we took special care to move each base that had a negative exponent, but the expression itself did not become negative. Also, it is important to remember that exponents only effect the base. The 2 in the denominator has an exponent of one (we don't always write it, but we know it's there), so it does not move with the d.

World Note

Nicolas Chuquet, the French mathematician of the 15^{th} century wrote $12^{1\bar{m}}$ to indicate $12x^{-1}$. This was the first known use of the negative exponent.

6.1.6 Properties of exponents

Putting all the rules together, we can simplify more complex expression containing exponents. Here we apply all the rules of exponents to simplify expressions.

General exponent rules

Let a and b be positive real numbers and n and m be any real numbers.

Rule 1. $a^n \cdot a^m = a^{n+m}$

Rule 2. $\dfrac{a^n}{a^m} = a^{n-m}$

Rule 3. $(a^n)^m = a^{nm}$

Rule 4. $(ab)^n = a^n \cdot b^n$

Rule 5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Rule 6. $a^0 = 1$

Rule 7. $a^{-n} = \dfrac{1}{a^n}$

Rule 8. $\dfrac{1}{a^{-n}} = a^n$

Rule 9. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

Example 6.17

Simplify: $\dfrac{4x^{-5}y^{-3} \cdot 3x^3y^{-2}}{6x^{-5}y^3}$

Solution.

$$\frac{4x^{-5}y^{-3} \cdot 3x^{3}y^{-2}}{6x^{-5}y^{3}} \quad \text{Simplify the numerator by applying the product rule}$$

$$\frac{12x^{-2}y^{-5}}{6x^{-5}y^{3}} \quad \text{Simplify by applying the quotient rule}$$

$$\frac{12}{6} \cdot x^{-2-(-5)}y^{-5-3)} \quad \text{Simplify}$$

$$2x^{3}y^{-8} \quad \text{Rewrite with only positive exponents}$$

$$\frac{2x^{3}}{y^{8}} \quad \text{Simplified expression}$$

Example 6.18

Simplify: $\dfrac{\left(3ab^{3}\right)^{-2} \cdot ab^{-3}}{2a^{-4}b^{0}}$

Solution.

$$\frac{\left(3ab^{3}\right)^{-2} \cdot a^{1}b^{-3}}{2a^{-4}b^{0}} \quad \text{Apply POP and zero power rule}$$

$$\frac{3^{-2}a^{-2}b^{-6} \cdot a^{1}b^{-3}}{2a^{-4} \cdot 1} \quad \text{Apply product rule}$$

$$\frac{3^{-2}a^{-1}b^{-9}}{2a^{-4}} \quad \text{Apply the quotient rule}$$

$$\frac{3^{-2}a^{3}b^{-9}}{2} \quad \text{Rewrite with only positive exponents}$$

$$\frac{a^{3}}{2 \cdot 3^{2} \cdot b^{9}} \quad \text{Simplify}$$

$$\frac{a^{3}}{18b^{9}} \quad \text{Simplified expression}$$

It is important to point out that when we simplified 3^{-2}, we moved the 3^{-2} to the denominator and the exponent became positive. We did not make the number negative. Negative exponents never make the bases negative; they simply mean we have to take the reciprocal of the base.

Example 6.19

Simplify: $\left(\dfrac{3x^{-2}y^{5}z^{3} \cdot 6x^{-6}y^{-2}z^{-3}}{9(x^{2}y^{-2})^{-3}}\right)^{-3}$

Solution.

This example looks more involved that any of the other examples, but we will apply the same method. It is advised, in these types of problems, that we simplify the expression inside the parenthesis first, and then apply the POP rule. We even should start with simplifying each numerator and denominator before simplifying the fraction with the quotient rule.

$$\begin{array}{rl}
\left(\dfrac{3x^{-2}y^5z^3 \cdot 6x^{-6}y^{-2}z^{-3}}{9(x^2y^{-2})^{-3}}\right)^{-3} & \text{Simplify each numerator and denominator} \\[2ex]
\left(\dfrac{18x^{-8}y^3z^0}{9x^{-6}y^6}\right)^{-3} & \text{Apply the quotient rule} \\[2ex]
(2x^{-2}y^{-3}z^0)^{-3} & \text{Apply the POP rule} \\[2ex]
2^{-3}x^{-6}y^9z^0 & \text{Rewrite only with positive exponents} \\
\dfrac{x^6y^9}{2^3} & \text{Simplify} \\[2ex]
\dfrac{x^6y^9}{8} & \text{Simplified expression}
\end{array}$$

6.1.7 Exponent Rules and Properties Homework

Simplify. Be sure to follow the simplifying rules and write answers with positive exponents.

1. $4 \cdot 4^4 \cdot 4^4$

2. $4 \cdot 2^2$

3. $3m \cdot 4mn$

4. $2m^4n^2 \cdot 4nm^2$

5. $(3^3)^4$

6. $(4^4)^2$

7. $(2u^3v^2)^2$

8. $(2a^4)^4$

9. $\dfrac{4^5}{4^3}$

10. $\dfrac{3^2}{3}$

11. $\dfrac{3nm^2}{3n}$

12. $\dfrac{4x^3y^4}{3xy^3}$

13. $(x^3y^4 \cdot 2x^2y^3)^2$

14. $2x(x^4y^4)^4$

15. $\dfrac{2x^7y^5}{3x^3y \cdot 4x^2y^3}$

16. $\left(\dfrac{(2x)^3}{x^3}\right)^2$

17. $\left(\dfrac{2y^{17}}{(2x^2y^4)^4}\right)^3$

18. $\left(\dfrac{2mn^4 \cdot 2m^4n^4}{mn^4}\right)^3$

19. $\dfrac{2xy^5 \cdot 2x^2y^3}{2xy^4 \cdot y^3}$

20. $\dfrac{q^3r^2 \cdot (2p^2q^2r^3)^2}{2p^3}$

21. $\left(\dfrac{zy^3 \cdot z^3x^4y^4}{x^3y^3z^3}\right)^4$

22. $\dfrac{2x^2y^2z^6 \cdot 2zx^2y^2}{(x^2z^3)^2}$

23. $4 \cdot 4^4 \cdot 4^2$

24. $3 \cdot 3^3 \cdot 3^2$

25. $3x \cdot 4x^2$

26. $x^2y^4 \cdot xy^2$

27. $(4^3)^4$

28. $(3^2)^3$

29. $(xy)^3$

30. $(2xy)^4$

31. $\dfrac{3^7}{3^3}$

32. $\dfrac{3^4}{3}$

33. $\dfrac{x^2y^4}{4xy}$

34. $\dfrac{xy^3}{4xy}$

35. $(u^2v^2 \cdot 2u^4)^3$

36. $\dfrac{3vu^5 \cdot 2v^3}{uv^2 \cdot 2u^3v}$

37. $\dfrac{2ba^7 \cdot 2b^4}{ba^2 \cdot 3a^3b^4}$

38. $\dfrac{2a^2b^2a^7}{(ba^4)^2}$

39. $\dfrac{yx^2 \cdot (y^4)^2}{2y^4}$

40. $\dfrac{n^3(n^4)^2}{2mn}$

41. $\dfrac{(2y^3x^2)^2}{2x^2y^4 \cdot x^2}$

42. $\dfrac{2x^4y^5 \cdot 2z^{10}x^2y^7}{(xy^2z^2)^4}$

43. $\left(\dfrac{2q^3p^3r^4 \cdot 2p^3}{(qrp^3)^2}\right)^4$

44. $2x^4y^{-2} \cdot (2xy^3)^4$

45. $(a^4b^{-3})^3 \cdot 2a^3b^{-2}$

46. $(2x^2y^2)^4x^{-4}$

47. $(x^3y^4)^3 \cdot x^{-4}y^4$

48. $\dfrac{2x^{-3}y^2}{3x^{-3}y^3 \cdot 3x^0}$

49. $\dfrac{4xy^{-3} \cdot x^{-4}y^0}{4y^{-1}}$

50. $\dfrac{u^2v^{-1}}{2u^0v^4 \cdot 2uv}$

51. $\dfrac{u^2}{4u^0v^3 \cdot 3v^2}$

52. $\dfrac{2y}{(x^0y^2)^4}$

53. $\left(\dfrac{2a^2b^3}{a^{-1}}\right)^4$

54. $\dfrac{2nm^4}{(2m^2n^2)^4}$

55. $\dfrac{(2mn)^4}{m^0n^{-2}}$

56. $\dfrac{y^3 \cdot x^{-3}y^2}{(x^4y^2)^3}$

57. $\dfrac{2u^{-2}v^3 \cdot (2uv^4)^{-1}}{2u^{-4}v^0}$

58. $\left(\dfrac{2x^0 \cdot y^4}{y^4}\right)^3$

59. $\dfrac{y(2x^4y^2)^2}{2x^4y^0}$

60. $\dfrac{2yzx^2}{2x^4y^4z^{-2} \cdot (zy^2)^4}$

61. $\dfrac{2kh^0 \cdot 2h^{-3}k^0}{(2kj^3)^2}$

62. $\dfrac{(cb^3)^2 \cdot 2a^{-3}b^2}{(a^3b^{-2}c^3)^3}$

63. $\dfrac{(yx^{-4}z^2)^{-1}}{z^3 \cdot x^2y^3z^{-1}}$

64. $2a^{-2}b^{-3} \cdot (2a^0b^4)^4$

65. $2x^3y^2 \cdot (2x^3)^0$

66. $(m^0n^3 \cdot 2m^{-3}n^{-3})^0$

67. $2m^{-1}n^{-3} \cdot (2m^{-1}n^{-3})^4$

68. $\dfrac{3y^3}{3yx^3 \cdot 2x^4y^{-3}}$

69. $\dfrac{3x^3y^2}{4y^{-2} \cdot 3x^{-2}y^{-4}}$

70. $\dfrac{2xy^2 \cdot 4x^3y^{-4}}{4x^{-4}y^{-4} \cdot 4x}$

71. $\dfrac{2x^{-2}y^2}{4yx^2}$

72. $\dfrac{(a^4)^4}{2b}$

73. $\left(\dfrac{2y^{-4}}{x^2}\right)^{-2}$

74. $\dfrac{2y^2}{(x^4y^0)^{-4}}$

75. $\dfrac{2x^{-3}}{(x^4y^{-3})^{-1}}$

76. $\dfrac{2x^{-2}y^0 \cdot 2xy^4}{(xy^0)^{-1}}$

77. $\dfrac{2yx^2 \cdot x^{-2}}{(2x^0y^4)^{-1}}$

78. $\dfrac{u^{-3}v^{-4}}{2v(2u^{-3}v^4)^0}$

79. $\dfrac{b^{-1}}{(2a^4b^0)^0 \cdot 2a^{-3}b^2}$

80. $\dfrac{2b^4c^{-2} \cdot (2b^3c^2)^{-4}}{a^{-2}b^4}$

81. $\left(\dfrac{(2x^{-3}y^0z^{-1})^3 \cdot x^{-3}y^2}{2x^3}\right)^{-2}$

82. $\dfrac{2q^4 \cdot m^2p^2q^4}{(2m^{-4}p^2)^3}$

83. $\dfrac{2mpn^{-3}}{(m^0n^{-4}p^2)^3 \cdot 2n^2p^0}$

6.2 Scientific notation

One application of exponent properties is *scientific notation*. Scientific notation is used to represent really large or really small numbers, like the numbers that are too large or small to display on the calculator. For example, the distance light travels per year in miles is a very large number (5,879,000,000,000) and the mass of a single hydrogen atom in grams is a very small number (0.00000000000000000000000167). Basic operations, such as multiplication and division, with these numbers, would be quite cumbersome. However, the exponent properties allow us for simpler calculations.

Definition

Scientific notation is a notation for representing extremely large or small numbers in form of

$$a \times 10^N,$$

where N is an integer, $1 \leq a < 10$, and N is number of decimal places from the right or left we moved to obtain a.

A few notes regarding scientific notation:

- N is the way we convert between scientific and standard notation.
- N represents the number of times we multiply or divide by 10. (Recall, multiplying by 10 moves the decimal point of a number one place value.)
- We decide which direction to move the decimal (left or right) by remembering that in standard notation, positive exponents are numbers greater than ten and negative exponents are numbers less than one (but larger than zero).

Case 1. If we move the decimal to the left with a number in standard notation, then N will be positive.

Case 2. If we move the decimal to the right with a number in standard notation, then N will be negative.

6.2.1 Convert numbers to scientific notation

Example 6.20

Convert 14,200 to scientific notation.

Solution.

Since this number is greater than 10, then we move the decimal to the left and N is positive. First we will find a, then N.

$$\begin{array}{rl} 14200.0 & \text{Identify the location of the decimal} \\ 1.\,4\;\;2\;\;0\;\;0.0 & \text{Four decimal places to the left} \\ 1.42 & \text{The value for } a \end{array}$$

Since we moved 4 decimal places to the left to obtain 1.42, then we know $N = 4$, i.e., the exponent on the 10 is 4. Hence, rewriting 14,200 from standard notation to scientific notation, we get

$$1.42 \times 10^4$$

Note

Be sure to always move the decimal however many decimal places to obtain a number between 1 and 10. In example 6.20, we only moved four decimal places because that is the number of decimal places we needed to move to obtain a number between 1 and 10.

Example 6.21

Convert 0.0042 to scientific notation.

Solution.

Since this number is less than 1 (but greater than zero), then we move the decimal to the right and N is negative. First we will find a, then N.

$$\begin{array}{rl} 0.0042 & \text{Identify the location of the decimal} \\ 0.\ 0\ \ 0\ \ 4\ .2 & \text{Three decimal places to the right} \\ 4.2 & \text{The value for } a \end{array}$$

Since we moved 3 decimal places to the right to obtain 4.2, then we know $N = -3$, i.e., the exponent on the 10 is -3. Hence, rewriting 0.0042 from standard notation to scientific notation, we get

$$4.2 \times 10^{-3}$$

6.2.2 Convert numbers from scientific notation to standard notation

Convert numbers from scientific notation to standard notation

To convert a number from scientific notation of the form

$$a \times 10^{N}$$

to standard notation, we can follow these rules of thumb.

- If N is positive, this means the original number was greater than 10, we move the decimal to the right N times.
- If N is negative, this means the original number was less than 1 (but greater than zero), we move the decimal to the left N times.

Example 6.22

Convert 3.21×10^{5} to standard notation.

Solution.

Since $N = 5$, which is positive, then this means the standard notation of the number is greater than 10 and we move the decimal to the right 5 times.

$$\begin{array}{rl} 3.21 & \text{Identify the location of the decimal} \\ 3.\ 2\ \ 1\ \ 0\ \ 0\ \ 0. & \text{Five decimal places to the right} \\ 321000. & \text{Standard notation} \end{array}$$

Since we moved 5 decimal places to the right to obtain 321,000, notice, as we were moving the decimal, there were place values with no digits and so we wrote in the zeros. In general, we do this when there are place values with no digits when expanding the numbers.

Example 6.23

Convert 7.4×10^{-3} to standard notation.

Solution.

Since $N = -3$, which is negative, then this means the standard notation of the number is less than 1 (but greater than zero) and we move the decimal to the left 3 times.

7.4	Identify the location of the decimal
$0.\,0\;\;0\;7.4$	Three decimal places to the left
0.0074	Standard notation

Since we moved 3 decimal places to the left to obtain 0.0074, notice we had to write zeros in for the tenths and hundredths place.

6.2.3 Multiply and divide numbers in scientific notation

Converting numbers between standard notation and scientific notation is important in understanding scientific notation and its purpose. Next, we multiply and divide numbers in scientific notation using the exponent properties. If the immediate result isn't written in scientific notation, we will complete an additional step in writing the answer in scientific notation.

Steps for multiplying and dividing numbers in scientific notation

Step 1. Rewrite the factors as multiplying or dividing a-values and then multiplying or dividing 10^N values.

Step 2. Multiply or divide the a values and apply the product or quotient rule of exponents to add or subtract the exponents, N, on the base 10s, respectively.

Step 3. Be sure the result is in scientific notation. If not, then rewrite in scientific notation.

Example 6.24

Multiply: $\left(2.1 \times 10^{-7}\right)\left(3.7 \times 10^{5}\right)$

Solution.

Step 1. Rewrite the factors as multiplying a-values and then multiplying 10^N values.

$$(2.1)(3.7) \times \left(10^{-7} \cdot 10^{5}\right)$$

Step 2. Multply a values and apply the product rule of exponents on the 10^N values.

$$\begin{aligned} (2.1)(3.7) \times 10^{-7+5} &\qquad \text{Simplify} \\ 7.77 \times 10^{-2} &\qquad \text{Product} \end{aligned}$$

Step 3. Since the product resulted in scientific notation, we leave it as is.

Example 6.25

Divide: $\dfrac{4.96 \times 10^4}{3.1 \times 10^{-3}}$

Solution.

Step 1. Rewrite the factors as dividing a-values and then dividing 10^N values.

$$\frac{4.96}{3.1} \times \frac{10^4}{10^{-3}}$$

Step 2. Multply a values and apply the quotient rule of exponents on the 10^N values.

$$\begin{array}{rl} \dfrac{4.96}{3.1} \times 10^{4-(-3)} & \text{Simplify} \\ 1.6 \times 10^7 & \text{Quotient} \end{array}$$

Step 3. Since the quotient resulted in scientific notation, we leave it as is.

Example 6.26

Multiply: $\left(4.7 \times 10^{-3}\right)\left(6.1 \times 10^9\right)$

Solution.

Step 1. Rewrite the factors as multiplying a-values and then multiplying 10^N values.

$$(4.7)(6.1) \times \left(10^{-3} \cdot 10^9\right)$$

Step 2. Multiply a values and apply the product rule of exponents on the 10^N values.

$$\begin{array}{rl} (4.7)(6.1) \times 10^{-3+9} & \text{Simplify} \\ 28.67 \times 10^6 & \text{Product} \end{array}$$

Step 3. Since the product resulted a number not in scientific notation, we have rewrite it so that it is in scientific notation. Hence, we need a to be a number at least 1 and less than 10, and 28.67 is greater than 10, then we move the decimal to the left and N is positive.

$$\begin{array}{rl} (28.67) \times 10^6 & \text{Identify the location of the decimal} \\ \left(2.8.67 \times 10^1\right) \times 10^6 & \text{One decimal place to the left} \\ 2.867 \times 10^1 \times 10^6 & \text{Apply product rule of exponents} \\ 2.867 \times 10^7 & \text{Scientific notation} \end{array}$$

World Note

Archimedes (287 BC-212 BC), the Greek mathematician, developed a system for representing large numbers using a system very similar to scientific notation. He used his system to calculate the number of grains of sand it would take to fill the universe. His conclusion was 10^{63} grains of sand because he figured the universe to have a diameter of 10^{14} stadia or about 2 light years.

Example 6.27

Divide: $\dfrac{2.014 \times 10^{-3}}{3.8 \times 10^{-7}}$

Solution.

Step 1. Rewrite the factors as dividing a-values and then dividing 10^N values.

$$\frac{2.014}{3.8} \times \frac{10^{-3}}{10^{-7}}$$

Step 2. Divide a values and apply the quotient rule of exponents on the 10^N values.

$$\begin{array}{rl} \dfrac{2.014}{3.8} \times 10^{-3-(-7)} & \text{Simplify} \\ 0.53 \times 10^{4} & \text{Quotient} \end{array}$$

Step 3. Since the quotient resulted a number not in scientific notation, we have rewrite it so that it is in scientific notation. Hence, we need a to be a number at least 1 and less than 10, and 0.53 is less than 1 (but greater than zero), then we move the decimal to the right and N is negative.

$$\begin{array}{rl} (0.53) \times 10^{4} & \text{Identify the location of the decimal} \\ \left(0.\,5\,.3 \times 10^{-1}\right) \times 10^{4} & \text{One decimal place to the right} \\ 5.3 \times 10^{-1} \times 10^{4} & \text{Apply product rule of exponents} \\ 5.3 \times 10^{3} & \text{Scientific notation} \end{array}$$

6.2.4 Scientific Notation Homework

Write each number in scientific notation

1. 885

2. 0.081

3. 0.039

4. 0.000744

5. 1.09

6. 15,000

Write each number in standard notation.

7. 8.7×10^5

8. 9×10^{-4}

9. 2×10^0

10. 2.56×10^2

11. 5×10^4

12. 6×10^{-5}

Simplify. Write each answer in scientific notation.

13. $(7 \times 10^{-1})(2 \times 10^{-3})$

14. $(5.26 \times 10^{-5})(3.16 \times 10^{-2})$

15. $(2.6 \times 10^{-2})(6 \times 10^{-2})$

16. $\dfrac{4.9 \times 10^1}{2.7 \times 10^{-3}}$

17. $\dfrac{5.33 \times 10^{-6}}{9.62 \times 10^{-2}}$

18. $(5.5 \times 10^{-5})^2$

19. $(7.8 \times 10^{-2})^5$

20. $(8.03 \times 10^4)^{-4}$

21. $\dfrac{6.1 \times 10^{-6}}{5.1 \times 10^{-4}}$

22. $(3.6 \times 10^0)(6.1 \times 10^{-3})$

23. $(1.8 \times 10^{-5})^{-3}$

24. $\dfrac{9 \times 10^4}{7.83 \times 10^{-2}}$

25. $\dfrac{3.22 \times 10^{-3}}{7 \times 10^{-6}}$

26. $\dfrac{2.4 \times 10^{-6}}{6.5 \times 10^0}$

27. $\dfrac{6 \times 10^3}{5.8 \times 10^{-3}}$

28. $(2 \times 10^{-6})(8.8 \times 10^{-5})$

29. $(5.1 \times 10^6)(9.84 \times 10^{-1})$

30. $\dfrac{7.4 \times 10^4}{1.7 \times 10^{-4}}$

31. $\dfrac{7.2 \times 10^{-1}}{7.32 \times 10^{-1}}$

32. $\dfrac{3.2 \times 10^{-3}}{5.02 \times 10^0}$

33. $(9.6 \times 10^3)^{-4}$

34. $(5.4 \times 10^6)^{-3}$

35. $(6.88 \times 10^{-4})(4.23 \times 10^1)$

36. $\dfrac{8.4 \times 10^5}{7 \times 10^{-2}}$

37. $(3.15 \times 10^3)(8 \times 10^{-1})$

38. $\dfrac{9.58 \times 10^{-2}}{1.14 \times 10^{-3}}$

39. $(8.3 \times 10^1)^5$

40. $\dfrac{5 \times 10^6}{6.69 \times 10^2}$

41. $(9 \times 10^{-2})^{-3}$

42. $(2 \times 10^4)(6 \times 10^1)$

6.3 Add and subtract polynomial expressions

Now that we have discussed exponent rules in great lengths, it is time to discuss *polynomials* and their operations.

Definition

- A ***monomial*** in one variable is the product of a coefficient and a variable raised to a positive integer exponent. A monomial is of the form

$$ax^k,$$

where a is the coefficient, x is the variable (and base), and k is the ***degree*** of the monomial. Recall, k is a positive integer.
- A ***binomial*** in one variable is the sum of two monomials.
- A ***polynomial*** is the sum or difference of monomials. The degree of a polynomial is the highest degree of all the terms in the polynomial.

Example 6.28

Rewrite the polynomial in standard form and identify the coefficients, variable terms, and degree of the polynomial

$$-12x^2 + x^3 - x + 2$$

Solution.

The ***standard form*** of a polynomial is where the polynomial is written with descending exponents:

$$x^3 - 12x^2 - x + 2$$

The coefficients are $1, -12, -1$, and 2; the variable terms are x^3, $-12x^2$, $-x$. The degree of the polynomial is 3 because that is the highest degree of all terms.

6.3.1 Evaluate polynomial expressions

If we are given a value for the variable in a polynomial, we can evaluate the polynomial.

Example 6.29

Evaluate $2x^2 - 4x + 6$ when $x = -4$.

Solution.

We plug-n-chug $x = -4$ for every x and simplify.

$$\begin{array}{rl}
2x^2 - 4x + 6 & \text{Plug-n-chug } x = -4 \\
2(-4)^2 - 4(-4) + 6 & \text{Evaluate} \\
32 + 16 + 6 & \text{Simplify} \\
54 & \text{Value of the polynomial when } x = -4
\end{array}$$

Note

It is important to be careful with negative variables, and exponents. Recall, the exponent is only applied to its base. For example, $-3^2 = -9$ because we evaluate 3^2 first, then multiply by a negative. On the other hand, $(-3)^2 = 9$ because we evaluate the entire base in parenthesis as $-3 \cdot -3 = 9$. In math, *if it looks different, then it is different.*

Example 6.30

Evaluate $-x^2 + 2x + 6$ when $x = 3$.

Solution.

$$\begin{aligned} -x^2 + 2x + 6 &\qquad \text{Plug-n-chug } x = 3 \\ -(3)^2 + 2(3) + 6 &\qquad \text{Evaluate} \\ -9 + 6 + 6 &\qquad \text{Simplify} \\ 3 &\qquad \text{Value of the polynomial when } x = 3 \end{aligned}$$

World Note

Ada Lovelace, in 1842, described Charles Babbage's Difference Engine that would be used to calculate values of polynomials. Her work became the foundation for what would become the modern computer (the programming language Ada was named in her honor) more than 100 years after her death from cancer.

6.3.2 Add and subtract polynomial expressions

Generally, when working with polynomials, we rarely know the value of the variable, so, next, we simplify polynomial expressions by adding and subtracting them. We will combine *like terms*.

Recall. Like terms are terms with the *same variable(s) as the base and exponent.*

Example 6.31

Add: $(4x^3 - 2x + 8) + (3x^3 - 9x^2 - 11)$

Solution.

We can add by combining like terms.

$$\begin{aligned} (4x^3 - 2x + 8) + (3x^3 - 9x^2 - 11) &\qquad \text{Rewrite without parenthesis} \\ 4x^3 - 2x + 8 + 3 + x^3 - 9x^2 - 11 &\qquad \text{Add like terms} \\ 7x^3 - 9x^2 - 2x - 3 &\qquad \text{Sum} \end{aligned}$$

Be sure to write the sum in standard form.

Example 6.32

Subtract: $(5x^2 - 2x + 7) - (3x^2 + 6x - 4)$

Solution.

$$\begin{aligned}(5x^2-2x+7)-(3x^2+6x-4) &\quad \text{Rewrite without parenthesis}\\ 5x^2-2x+7-3x^2-6x+4 &\quad \text{Subtract like terms}\\ 2x^2-8x+11 &\quad \text{Difference}\end{aligned}$$

Example 6.33

Simplify: $(2x^2-4x+3)+(5x^2-6x+1)-(x^2-9x+8)$

Solution.

$$\begin{aligned}(2x^2-4x+3)+(5x^2-6x+1)-(x^2-9x+8) &\quad \text{Rewrite without parenthesis}\\ 2x^2-4x+3+5x^2-6x+1-x^2+9x-8 &\quad \text{Combine like terms}\\ 6x^2-x-4 &\quad \text{Simplified expression}\end{aligned}$$

6.3.3 Add and Subtract Polynomial Expressions Homework

Evaluate the expression for the given value.

1. $-a^3 - a^2 + 6a - 21$ when $a = -4$

2. $n^2 + 3n - 11$ when $n = -6$

3. $n^3 - 7n^2 + 15n - 20$ when $n = 2$

4. $n^3 - 9n^2 + 23n - 21$ when $n = 5$

5. $-5n^4 - 11n^3 - 9n^2 - n - 5$ when $n = -1$

6. $x^4 - 5x^3 - x + 13$ when $x = 5$

7. $x^2 + 9x + 23$ when $x = -3$

8. $-6x^3 + 41x^2 - 32x + 11$ when $x = 6$

9. $x^4 - 6x^3 + x^2 - 24$ when $x = 6$

10. $m^4 + 8m^3 + 14m^2 + 13m + 5$ when $m = -6$

Simplify. Write the answer in standard form.

11. $(5p - 5p^4) - (8p - 8p^4)$

12. $(7m^2 + 5m^3) - (6m^3 - 5m^2)$

13. $(3n^2 + n^3) - (2n^3 - 7n^2)$

14. $(x^2 + 5x^3) + (7x^2 + 3x^3)$

15. $(8n + n^4) - (3n - 4n^4)$

16. $(3v^4 + 1) + (5 - v^4)$

17. $(1 + 5p^3) - (1 - 8p^3)$

18. $(6x^3 + 5x) - (8x + 6x^3)$

19. $(5n^4 + 6n^3) + (8 - 3n^3 - 5n^4)$

20. $(8x^2 + 1) - (6 - x^2 - x^4)$

21. $(3 + b^4) + (7 + 2b + b^4)$

22. $(1 + 6r^2) + (6r^2 - 2 - 3r^4)$

23. $(8x^3 + 1) - (5x^4 - 6x^3 + 2)$

24. $(4n^4 + 6) - (4n - 1 - n^4)$

25. $(2a + 2a^4) - (3a^2 - 5a^4 + 4a)$

26. $(6v + 8v^3) + (3 + 4v^3 - 3v)$

27. $(4p^2 - 3 - 2p) - (3p^2 - 6p + 3)$

28. $(7 + 4m + 8m^4) - (5m^4 + 1 + 6m)$

29. $(4b^3 + 7b^2 - 3) + (8 + 5b^2 + b^3)$

30. $(7n + 1 - 8n^4) - (3n + 7n^4 + 7)$

31. $(3 + 2n^2 + 4n^4) + (n^3 - 7n^2 - 4n^4)$

32. $(7x^2 + 2x^4 + 7x^3) + (6x^3 - 8x^4 - 7x^2)$

33. $(n - 5n^4 + 7) + (n^2 - 7n^4 - n)$

34. $(8x^2 + 2x^4 + 7x^3) + (7x^4 - 7x^3 + 2x^2)$

35. $(8r^4 - 5r^3 + 5r^2) + (2r^2 + 2r^3 - 7r^4 + 1)$

36. $(4x^3 + x - 7x^2) + (x^2 - 8 + 2x + 6x^3)$

37. $(2n^2 + 7n^4 - 2) + (2 + 2n^3 + 4n^2 + 2n^4)$

38. $(7b^3 - 4b + 4b^4) - (8b^3 - 4b^2 + 2b^4 - 8b)$

39. $(8 - b + 7b^3) - (3b^4 + 7b - 8 + 7b^2) + (3 - 3b + 6b^3)$

40. $(1 - 3n^4 - 8n^3) + (7n^4 + 2 - 6n^2 + 3n^3) + (4n^3 + 8n^4 + 7)$

41. $(8x^4 + 2x^3 + 2x) + (2x + 2 - 2x^3 - x^4) - (x^3 + 5x^4 + 8x)$

42. $(6x - 5x^4 - 4x^2) - (2x - 7x^2 - 4x^4 - 8) - (8 - 6x^2 - 4x^4)$

6.4 Multiply polynomial expressions

We can multiply polynomials of different forms, but the method will be the same. We first look at multiplying monomials, multiplying a monomial and polynomial, and then finish with multiplying polynomials. We are using the product rule of exponents and the distributive property.

6.4.1 Multiply a polynomial expression by a monomial

Example 6.34

Multiply: $(4x^3y^4z)(2x^2y^6z^3)$

Solution.

$(4x^3y^4z)(2x^2y^6z^3)$	Rewrite without parenthesis
$4x^3y^4z \cdot 2x^2y^6z^3$	Multiply coefficients and apply the product rule of exponents
$4 \cdot 2 \cdot x^{3+2}y^{4+6}z^{1+3}$	Simplify
$8x^5y^{10}z^4$	Product

Example 6.35

Multiply: $4x^3(5x^2 - 2x + 5)$

Solution.

$4x^3(5x^2 - 2x + 5)$	Distribute $4x^3$
$4x^3 \cdot 5x^2 - 4x^3 \cdot 2x + 4x^3 \cdot 5$	Multiply and apply the product rule of exponents
$20x^5 - 8x^4 + 20x^3$	Product

Example 6.36

Multiply: $2a^3b(3ab^2 - 4a)$

Solution.

$2a^3b(3ab^2 - 4a)$	Distribute $2a^3b$
$2a^3b \cdot 3ab^2 - 2a^3b \cdot 4a$	Multiply and apply the product rule of exponents
$6a^{3+1}b^{1+2} - 8a^{3+1}b$	Simplify
$6a^4b^3 - 8a^4b$	Product

6.4.2 Multiplying with binomials

There are several different methods for multiplying polynomials, all of which result in the same answer. We discuss multiplying by distribution and the FOIL method.

Example 6.37

Multiply: $(3x + 5)(x + 13)$

Solution.

We will multiply using distribution and then simplify.

$$\begin{array}{rl} (3x+5)(x+13) & \text{Distribute } 3x \text{ and } 5 \text{ to } (x+13) \\ {\color{blue}3x}(x+13)+{\color{blue}5}(x+13) & \text{Distribute} \\ 3x^2+39x+5x+65 & \text{Combine like terms} \\ 3x^2+44x+65 & \text{Product} \end{array}$$

Example 6.38

Multiply: $(4x+7y)(3x-2y)$

Solution.

We will multiply using the FOIL method. FOIL is an acronym and represents

First	-Multiply the first terms in each parenthesis
Outer	-Multiply the outer terms in each parenthesis
Inner	-Multiply the inner terms in each parenthesis
Last	-Multiply the last terms in each parenthesis

$$\begin{aligned} (4x+7y)(3x-2y) &= \underbrace{(4x)(3x)}_{F} - \underbrace{(4x)(2y)}_{O} + \underbrace{(7y)(3x)}_{I} - \underbrace{(7y)(2y)}_{L} \\ &= 12x^2 - 8xy + 21xy - 14y^2 \\ &= 12x^2 + 13xy + 14y^2 \end{aligned}$$

6.4.3 Multiplying with trinomials

A ***trinomial*** is a polynomial with three terms. Usually, in Algebra, a trinomial takes the form of ax^2+bx+c, where a, b, and c are coefficients.

Example 6.39

Multiply: $(2x-5)\left(4x^2-7x+3\right)$

Solution.

Since we are multiplying a binomial with a trinomial, we can use distribution to multiply.

$$\begin{array}{rl} (2x-5)\left(4x^2-7x+3\right) & \text{Distribute } 2x \text{ and } -5 \text{ to } \left(4x^2-7x+3\right) \\ {\color{blue}2x}\cdot\left(4x^2-7x+3\right)-{\color{blue}5}\left(4x^2-7x+3\right) & \text{Distribute} \\ 8x^3-14x^2+6x-20x^2+35x-15 & \text{Combine like terms} \\ 8x^3-34x^2+41x-15 & \text{Product} \end{array}$$

Example 6.40

Multiply: $\left(5x^2+x-10\right)\left(3x^2-10x-6\right)$

Solution.

Since we are multiplying a trinomial with a trinomial, then we can use distribution to multiply.

$$\begin{array}{rl} \left(5x^2+x-10\right)\left(3x^2-10x-6\right) & \text{Distribute } 5x^2, x, \text{ and } -10 \\ & \text{to } \left(3x^2-10x-6\right) \\ 5x^2\cdot\left(3x^2-10x-6\right)+x\left(3x^2-10x-6\right)-10\left(3x^2-10x-6\right) & \text{Distribute} \\ 15x^4-50x^3-30x^2+3x^3-10x^2-6x-30x^2+100x+60 & \text{Combine like terms} \\ 15x^4-47x^3-70x^2+94x+60 & \text{Product} \end{array}$$

6.4.4 Multiplying monomials and binomials

Example 6.41

Multiply: $3x(2x-4)(x+5)$

Solution.

We first use FOIL to multiply the binomials and then distribute the $3x$.

$$\begin{aligned} 3x(2x-4)(x+5) &= 3x\big(\underbrace{(2x)(x)}_{F}-\underbrace{(2x)(5)}_{O}-\underbrace{(4)(x)}_{I}-\underbrace{(4)(5)}_{L}\big) \\ &= 3x\left(2x^2+10x-4x-20\right) \\ &= 3x\left(2x^2+6x-20\right) \end{aligned}$$

Lastly, we distribute $3x$:

$$\begin{array}{l} 3x\left(2x^2+6x-20\right) \\ 3x\cdot 2x^2+3x\cdot 6x-3x\cdot 20 \\ 6x^3+18x^2-60x \end{array}$$

Thus, the product is $6x^3+18x^2-60x$.

☠ *Warning!*

In example 6.41, a common error is distributing the $3x$ first and into both parenthesis. While we can distribute the $3x$ into the $(2x-4)$ factor, we cannot distribute into both factors. We recommend to multiply the binomials first, then distribute the monomial last.

6.4.5 Multiplying polynomial functions

We can multiply two polynomial functions the same way we multiply polynomial expressions, except, now, we have functions. The method is the same, but the notation and problems change.

Definition

If f and g are two functions of x, then

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

where x is in the domain of f and g.

Example 6.42

Let $f(x) = 2x - 1$ and $g(x) = x + 4$. Find $(f \cdot g)(x)$.

Solution.

We start by applying the definition, then simplify completely.

$$\begin{aligned} (f \cdot g)(x) &= f(x) \cdot g(x) && \text{Apply the definition} \\ (f \cdot g)(x) &= (2x - 1) \cdot (x + 4) && \text{Multiply two binomials} \\ (f \cdot g)(x) &= 2x^2 + 8x - x - 4 && \text{Combine like terms} \\ (f \cdot g)(x) &= 2x^2 + 7x - 4 && \text{The product of } f \text{ and } g \end{aligned}$$

6.4.6 Multiply Polynomial Expressions Homework

Multiply and simplify.

1. $6(p-7)$

2. $2(6x+3)$

3. $5m^4(4m+4)$

4. $(4n+6)(8n+8)$

5. $(8b+3)(7b-5)$

6. $(4x+5)(2x+3)$

7. $(3v-4)(5v-2)$

8. $(6x-7)(4x+1)$

9. $(5x+y)(6x-4y)$

10. $(x+3y)(3x+4y)$

11. $(7x+5y)(8x+3y)$

12. $(r-7)(6r^2-r+5)$

13. $(6n-4)(2n^2-2n+5)$

14. $(6x+3y)(6x^2-7xy+4y^2)$

15. $(8n^2+4n+6)(6n^2-5n+6)$

16. $(5k^2+3k+3)(3k^2+3k+6)$

17. $3(3x-4)(2x+1)$

18. $3(2x+1)(4x-5)$

19. $7(x-5)(x-2)$

20. $6(4x-1)(4x+1)$

21. $4k(8k+4)$

22. $3n^2(6n+7)$

23. $3(4r-7)$

24. $(2x+1)(x-4)$

25. $(r+8)(4r+8)$

26. $(7n-6)(n+7)$

27. $(6a+4)(a-8)$

28. $(5x-6)(4x-1)$

29. $(2u+3v)(8u-7v)$

30. $(8u+6v)(5u-8v)$

31. $(5a+8b)(a-3b)$

32. $(4x+8)(4x^2+3x+5)$

33. $(2b-3)(4b^2+4b+4)$

34. $(3m-2n)(7m^2+6mn+4n^2)$

35. $(2a^2+6a+3)(7a^2-6a+1)$

36. $(7u^2+8uv-6v^2)(6u^2+4uv+3v^2)$

37. $5(x-4)(2x-3)$

38. $2x(4x+1)(2x-6)$

39. $5x(2x-1)(4x+1)$

40. $3x^2(2x+3)(6x+9)$

Perform the indicated operations given the set of functions.

41. Let $g(x)=4x+5$ and $h(x)=x^2+5x$, find $(g\cdot h)(x)$.

42. Let $p(t)=t-4$ and $r(t)=2t$, find $(p\cdot r)(t)$.

43. Let $f(n)=-2n^2-5n$ and $k(n)=n+5$, find $(f\cdot k)(n)$.

6.5 Special products

There are a few shortcuts that we can take when multiplying polynomials. If we can recognize when to use them, we should so that we can obtain the results even quicker. In future chapters, we will need to be efficient in these techniques since multiplying polynomials will only be one of the steps in the problem.

6.5.1 Difference of two squares

The first shortcut is called a *difference of two squares*. A sum and a difference is easily recognized as the numbers and variables are exactly the same, but the signs in the middle are opposites.

Example 6.43

Multiply: $(a+b)(a-b)$

Solution.

We can multiply these binomials by distribution.

$$\begin{aligned}
(a+b)(a-b) &\quad \text{Distribute } a \text{ and } b \text{ to } (a-b)\\
a(a-b)+b(a-b) &\quad \text{Distribute}\\
a^2-ab+ba-b^2 &\quad \text{Combine like terms}\\
a^2\cancel{-ab+ba}-b^2 &\quad \text{Simplify}\\
a^2-b^2 &\quad \text{Product}
\end{aligned}$$

Notice the middle terms cancelled and the product is a *difference of two squares*: a^2-b^2.

Difference of two squares

Given a product of two binomials, where the terms are the same but opposite middle signs, the product results in a difference of two squares, the squares of the terms:

$$(a+b)(a-b)=a^2-b^2$$

So awesome, right? This means if we are given any product of two binomials of this form, we can just square the terms and put a subtraction sign in between. Let's look at a couple of examples.

Example 6.44

Multiply: $(3x+7)(3x-7)$

Solution.

Notice the terms are $3x$ and 7 and have opposite middle signs. Hence, we can use the difference of two squares formula to arrive at the product quickly:

$$\begin{aligned}
(3x+7)(3x-7) &\quad \text{Terms are } 3x \text{ and } 7\\
(3x)^2-(7)^2 &\quad \text{Square the terms and put a subtraction sign in between}\\
9x^2-49 &\quad \text{Product}
\end{aligned}$$

Example 6.45

Multiply: $(2x-6y)(2x+6y)$

Solution.

Notice the terms are $2x$ and $6y$ and have opposite middle signs. Hence, we can use the difference of two squares formula to arrive at the product quickly:

$(2x - 6y)(2x + 6y)$	Terms are $2x$ and $6y$
$(2x)^2 - (6y)^2$	Square the terms and put a subtraction sign in between
$4x^2 - 36y^2$	Product

Warning!

It is interesting to note that while we can obtain a product like $a^2 - b^2$, it is *impossible* to obtain a product like $a^2 + b^2$. There are no two binomials in the real number system in which are multiplied to obtain a sum of two squares. Do not be fooled, though. There are products of two binomials out in the world that will result in a sum of two squares, but just not in this course.

6.5.2 Perfect square trinomials

Another shortcut used to multiply binomials is called *perfect square trinomials*. These are easy to recognize because this product is the square of a binomial. Let's take a look at an example.

Example 6.46

Multiply: $(a + b)^2$

Solution.

We can multiply these binomials by distribution.

$(a + b)^2$	Rewrite as a product of two binomials
$(a + b)(a + b)$	Distribute a and b to $(a + b)$
$a(a + b) + b(a + b)$	Distribute
$a^2 + ab + ba + b^2$	Combine like terms
$a^2 + 2ab + b^2$	Product

Notice the first term is the square of a, the middle term is 2 times the product of a and b, and the last term is the square of b. I.e., square the first, twice the product, square the last. Hence, the square of a binomial is a *perfect square trinomial*.

Perfect square trinomial

Given a square of a binomial, where the terms are the same but can have addition or subtraction middle signs, the product results in a perfect square trinomial:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

Example 6.47

Simplify: $(x - 5)^2$

Solution.

Notice this is the square of binomial $(x-5)$. We can use the perfect square trinomial formula to simplify.

$(x-5)^2$	Terms are x and 5
$(x)^2 - 2(x)(5) + (5)^2$	Follow the formula for $(a-b)^2$
$x^2 - 10x + 25$	Product

Example 6.48

Simplify: $(2x+9)^2$

Solution.

Notice this is the square of binomial $(2x+9)$. We can use the perfect square trinomial formula to simplify.

$(2x+9)^2$	Terms are $2x$ and 9
$(2x)^2 + 2(2x)(9) + (9)^2$	Follow the formula for $(a+b)^2$
$4x^2 + 36x + 81$	Product

Example 6.49

Simplify: $(3x-7y)^2$

Solution.

Notice this is the square of binomial $(3x-7y)$. We can use the perfect square trinomial formula to simplify.

$(3x-7y)^2$	Terms are $3x$ and $7y$
$(3x)^2 - 2(3x)(7y) + (7y)^2$	Follow the formula for $(a-b)^2$
$9x^2 - 42x + 49y^2$	Product

Warning!

Be *very careful* when we are squaring a binomial. Be sure to avoid the common error of only squaring the first and last terms. A common error is to do the following: $(x-5)^2 = x^2 - 25$ (or $x^2 + 25$). Notice both of these are missing the middle term, $-10x$.

Another important observation is that the middle term in the answer always has the same sign as the middle term in the given problem.

These two formulas are important to commit to memory. The more familiar we are with them, the next two chapters will be so much easier. The final example covers both types of problems. Be sure to notice the difference between the examples.

Example 6.50

Let's take a look at three examples side-by-side to see the difference between all the formulas. Let's multiply

$$(4x-7)(4x+7), (4x+7)^2, (4x-7)^2$$

Solution.

We apply the formulas to simplify each product.

$(4x-7)(4x+7)$	$(4x+7)^2$	$(4x-7)^2$
$(4x)^2-(7)^2$	$(4x)^2+2(4x)(7)+(7)^2$	$(4x)^2-2(4x)(7)+(7)^2$
$16x^2-49$	$16x^2+56x+49$	$16x^2-56x+49$

We see that the first product is a difference of two squares and the product is two terms. The second and third products are squares of binomials that results in perfect square trinomials and are three terms each.

World Note

There are also formulas for higher powers of binomials as well, such as $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$. While French mathematician, *Blaise Pascal*, often gets credit for working with these expansions of binomials in the 17^{th} century, Chinese mathematicians had been working with them almost 400 years earlier.

6.5.3 Special Products Homework

Find each product by applying the special products formulas.

1. $(x+8)(x-8)$

2. $(1+3p)(1-3p)$

3. $(1-7n)(1+7n)$

4. $(5n-8)(5n+8)$

5. $(4x+8)(4x-8)$

6. $(4y-x)(4y+x)$

7. $(4m-8n)(4m+8n)$

8. $(6x-2y)(6x+2y)$

9. $(a+5)^2$

10. $(x-8)^2$

11. $(p+7)^2$

12. $(7-5n)^2$

13. $(5m-8)^2$

14. $(5x+7y)^2$

15. $(2x+2y)^2$

16. $(5+2r)^2$

17. $(2+5x)^2$

18. $(4v-7)(4v+7)$

19. $(n-5)(n+5)$

20. $(4k+2)^2$

21. $(a-4)(a+4)$

22. $(x-3)(x+3)$

23. $(8m+5)(8m-5)$

24. $(2r+3)(2r-3)$

25. $(b-7)(b+7)$

26. $(7a+7b)(7a-7b)$

27. $(3y-3x)(3y+3x)$

28. $(1+5n)^2$

29. $(v+4)^2$

30. $(1-6n)^2$

31. $(7k-7)^2$

32. $(4x-5)^2$

33. $(3a+3b)^2$

34. $(4m-n)^2$

35. $(8x+5y)^2$

36. $(m-7)^2$

37. $(8n+7)(8n-7)$

38. $(b+4)(b-4)$

39. $(7x+7)^2$

40. $(3a-8)(3a+8)$

6.6 Polynomial division

Dividing polynomials is a process very similar to long division of whole numbers. Before we look at long division with polynomials, we will first master dividing a polynomial by a monomial.

6.6.1 Polynomial division with monomials

We divide a polynomial by a monomial by rewriting the expression as separated fractions rather than one fraction. We use the fact

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

Example 6.51

Divide: $\dfrac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2}$

Solution.

Notice we have four terms with the same denominator. We can rewrite this expression as 4 fractions with the same denominator and then simplify.

$\dfrac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2}$	Rewrite as four fractions with the same denominator
$\dfrac{9x^5}{3x^2} + \dfrac{6x^4}{3x^2} - \dfrac{18x^3}{3x^2} - \dfrac{24x^2}{3x^2}$	Reduce and apply the quotient rule of exponents
$3x^3 + 2x^2 - 6x - 8$	Quotient

Example 6.52

Divide: $\dfrac{8x^3 + 4x^2 - 2x + 6}{4x^2}$

Solution.

Notice we have four terms with the same denominator. We can rewrite this expression as 4 fractions with the same denominator and then simplify.

$\dfrac{8x^3 + 4x^2 - 2x + 6}{4x^2}$	Rewrite as four fractions with the same denominator
$\dfrac{8x^3}{4x^2} + \dfrac{4x^2}{4x^2} - \dfrac{2x}{4x^2} + \dfrac{6}{4x^2}$	Reduce and apply the quotient rule of exponents
$2x + 1 - \dfrac{1}{2}x^{-1} + \dfrac{3}{2}x^{-2}$	Rewrite with positive exponents
$2x + 1 - \dfrac{1}{2x} + \dfrac{3}{2x^2}$	Quotient

Notice that sometimes we have fractions in the quotient. As long as the fractions are reduced, it is correct. Also, the second term $\dfrac{4x^2}{4x^2}$ reduced completely to one.

6.6.2 Polynomial division with polynomials

Long division is required when we divide by a polynomial, i.e., when there is a sum or difference of terms in the denominator. Long division with polynomials works similar to long division with whole numbers. Let's review an example.

Example 6.53

Divide: $631 \div 4$

Solution.

Let's review this example. The divisor is 4 and the dividend is 631. The answer is called the quotient. First we rewrite the division with the divisor on the outside, then the long division symbol, and the dividend inside the long division symbol.

$\ 1\ \ 5\ \ 7$	
$4\overline{)\,6\ \ 3\ \ 1}$	How many times does 4 divide into 6?
-4	Once. We write 1 over the 6, keeping place values.
$2\ \ 3$	Bring down the next place value, 3, and how many times does 4 divide into 23?
$-2\ \ 0$	5 times. We write 5 over the 3, keeping place values.
$3\ \ 1$	
$-\ \ 2\ \ 8$	Bring down the next place value, 1, and how many times does 4 divide into 31?
3	7 times. We write 7 over the 1, keeping place values.

Hence, 3 is the remainder. So, we write the answer as the quotient, plus the remainder as a fraction:

$$157 + \frac{3}{4}$$

Simplifying this sum, we write $157\frac{3}{4}$.

This method may seem elementary, but it isn't the arithmetic we want to review, it is the method. We use the same method as we did in arithmetic, but now with polynomials.

Note

When writing the answer with a remainder, we write the answer as

$$\text{quotient } + \frac{\text{remainder}}{\text{divisor}}$$

Recall, if we are given $a \div b = c$, then

$$\underbrace{a}_{\text{dividend}} \div \underbrace{b}_{\text{divisor}} = \underbrace{c}_{\text{quotient}}$$

Example 6.54

Divide: $\dfrac{3x^3 - 5x^2 - 32x + 7}{x - 4}$

Solution.

Let's start by writing the division as long division:

$$x-4\overline{\big)\ 3x^3 \quad -5x^2 \quad -32x \quad +7}$$

Now, we follow the same method as we did for arithmetic. Be sure to keep place values and change the signs for subtracting.

$$\begin{array}{rrrrrl}
 & & 3x^2 & +7x & -4 & \\
\cline{2-5}
x-4\big) & 3x^3 & -5x^2 & -32x & +7 & \text{How many times does } x \text{ divide into } 3x^3\text{?} \\
- & \big(3x^3 & -12x^2\big) & & & \text{Multiply } 3x^2 \text{ and } (x-4),\ \text{then subtract.} \\
\cline{2-5}
 & & 7x^2 & -32x & & \text{Bring down the next place value} \\
 & & -\big(7x^2 & -28x\big) & & \text{How many times does } x \text{ divide into } 7x^2\text{? } 7x \\
\cline{3-5}
 & & & -4x & +7 & \text{Bring down the next place value} \\
 & & & -\big(-4x & +16\big) & \text{How many times does } x \text{ divide into } -4x\text{? } -4 \\
\cline{4-5}
 & & & & -9 & \text{Remainder}
\end{array}$$

We can see now that the method just repeats itself until we obtain a value that the divisor doesn't divide into and we obtain a remainder. Since the remainder is -9, then we have

$$3x^2 + 7x - 4 - \frac{9}{x-4}$$

Example 6.55

Divide: $\dfrac{6x^3 - 8x^2 + 10x + 103}{2x+4}$

Solution.

Following the same pattern as before, we rewrite the division as long division and then complete the long division process. However, in this example, we will," Draw a line and change the signs," so that we distribute the subtraction right away.

$$\begin{array}{rrrrr}
 & & 3x^2 & -10x & +25 \\
\cline{2-5}
2x+4\big) & 6x^3 & -8x^2 & +10x & +103 \\
 & -6x^3 & -12x^2 & & \\
\cline{2-5}
 & & -20x^2 & +10x & \\
 & & 20x^2 & +40x & \\
\cline{2-5}
 & & & 50x & +103 \\
 & & & -50x & -100 \\
\cline{3-5}
 & & & & 3
\end{array}$$

Since the remainder is 3, then we have

$$3x^2 - 10x + 25 + \frac{3}{2x+4}$$

Note

In example 6.55, instead of writing the subtraction sign with each step, we changed the terms to its opposite since that is essentially what we did in the previous example. We say,“ Draw a line, and change the signs,” for drawing the horizontal bar and the subtraction. This way, we can just add vertically. It is up to the discretion of the student which way to subtract, either subtract directly, or add the opposites.

6.6.3 Polynomial division with missing terms

Sometimes when dividing with polynomials, there may be a missing term in the dividend. We do not ignore the term, we just write in 0 as the coefficient.

Example 6.56

Divide: $\dfrac{2x^3 - 4x + 42}{x+3}$

Solution.

We rewrite the division as long division and follow the same method, but put in zero for the coefficient. In this case, we are missing the x^2 term; hence, we will put $0x^2$ for that term and then divide as usual.

$$
\begin{array}{r|rrrr}
 & & 2x^2 & -6x & +14 \\ \cline{2-5}
x+3 & 2x^3 & +0x^2 & -4x & +42 \\
 & -2x^3 & -6x^2 & & \\ \cline{2-5}
 & & -6x^2 & -4x & \\
 & & 6x^2 & +18x & \\ \cline{3-5}
 & & & 14x & +42 \\
 & & & -14x & -42 \\ \cline{4-5}
 & & & & 0
\end{array}
$$

Since the remainder is 0, then we have

$$2x^2 - 6x + 14$$

It is important to take a moment to check each problem to verify that the exponents descend and there are no missing terms. If so, we will have to adjust the problem as we did in example 6.56. Also, this final example illustrates, just as in regular long division, sometimes we have no remainder in a long division problem.

6.6.4 Polynomial division with functions

We can divide two polynomial functions the same way we divide polynomial expressions, except, now, we have functions. The method is the same, but the notation and problems change.

Definition

If f and g are two functions of x, then

$$(f \div g)(x) = \frac{f(x)}{g(x)}$$

where x is in the domain of f and g, and $g(x) \neq 0$.

Example 6.57

Let $f(x) = x^2 - 4x - 5$ and $g(x) = x - 5$. Find $(f \div g)(x)$.

Solution.

We start by applying the definition, then simplify completely.

$$(f \div g)(x) = \frac{f(x)}{g(x)}$$

$$(f \div g)(x) = \frac{x^2 - 4x - 5}{x - 5}$$

Taking the divisor $x - 5$, and the dividend to be $x^2 - 4x - 5$, we get

$$\begin{array}{r|rrr}
 & & x & +1 \\ \cline{2-4}
x-5 & x^2 & -4x & -5 \\
 & -x^2 & +5x & \\ \cline{2-4}
 & & x & -5 \\
 & & -x & +5 \\ \cline{3-4}
 & & & 0
\end{array}$$

Since there is no remainder, then $(f \div g)(x) = x + 1$.

6.6.5 Polynomial division by synthetic division

Another way to divide polynomials given a binomial divisor of the form $x - a$ is using solely the coefficients of the polynomials. ***Synthetic division*** is a method for dividing polynomials without using the given variables of the polynomials, but only the coefficients.

World Note

Paolo Ruffini was an Italian mathematician of the early 19$^{\text{th}}$ century. In 1809, he was the first to describe a process called synthetic division, which could also be used to divide polynomials. It is also called division through Ruffini's rule.

Example 6.58

Let's take Example 6.54, and apply synthetic division to obtain the same results as we did with polynomial division. Divide:

$$\frac{3x^3 - 5x^2 - 32x + 7}{x - 4}$$

Solution.

First we take the excluded value of the expression, which is when $x = 4$. We put this in the top right corner of the *synthetic division table*:

$$\begin{array}{r|rrrr} 4 & & & & \\ & & & & \\ \hline & & & & \end{array}$$

Then, following the excluded value, we place the coefficients of the dividend in the same top row, in standard order:

$$\begin{array}{r|rrrr} 4 & 3 & -5 & -32 & 7 \\ & & & & \\ \hline & & & & \end{array}$$

Bring the leading coefficient down to the bottom row:

$$\begin{array}{r|rrrr} 4 & 3 & -5 & -32 & 7 \\ & \downarrow & & & \\ \hline & 3 & & & \end{array}$$

Multiply the excluded value $x = 4$ with the leading coefficient, i.e, $4 \cdot 3 = 12$, and put the product under the second coefficient:

$$\begin{array}{r|rrr|r} 4 & 3 & -5 & -32 & 7 \\ & \downarrow & 12 & & \\ \hline & 3 & & & \end{array}$$

Add down the coefficient and the product, i.e., add $-5 + 12 = 7$, and place on the bottom row next to the leading coefficient:

$$\begin{array}{r|rrr|r} 4 & 3 & -5 & -32 & 7 \\ & \downarrow & 12 & & \\ \hline & 3 & 7 & & \end{array}$$

Multiply the excluded value $x = 4$ with the 7, i.e, $4 \cdot 7 = 28$, and put the product under the third coefficient:

$$\begin{array}{r|rrr|r} 4 & 3 & -5 & -32 & 7 \\ & \downarrow & 12 & 28 & \\ \hline & 3 & 7 & & \end{array}$$

Add down the coefficient and the product, i.e., add $-32 + 28 = -4$, and place on the bottom row next to the 7:

$$\begin{array}{r|rrr|r} 4 & 3 & -5 & -32 & 7 \\ & \downarrow & 12 & 28 & \\ \hline & 3 & 7 & -4 & \end{array}$$

Multiply the excluded value $x = 4$ with the -4, i.e, $4 \cdot -4 = -16$, and put the product under the fourth coefficient:

$$\begin{array}{r|rrr|r} 4 & 3 & -5 & -32 & 7 \\ & \downarrow & 12 & 28 & -16 \\ \hline & 3 & 7 & -4 & \end{array}$$

Add down the coefficient and the product, i.e., add $7 + (-16) = -9$, and place on the bottom row next to the 4:

$$\begin{array}{r|rrr|r} 4 & 3 & -5 & -32 & 7 \\ & \downarrow & 12 & 28 & -16 \\ \hline & 3 & 7 & -4 & -9 \end{array}$$

The first three numbers in the last row of our *synthetic division table* are the coefficients of the quotient polynomial. Remember, we started with a third-degree polynomial and divided by a first-degree polynomial, so the *quotient* is a second-degree polynomial. Hence, the quotient is $3x^2+7x-4$. The number in the bottom far-right corner, -9, is the remainder. Thus, the answer, written with the quotient and remainder, is

$$3x^2 + 7x - 4 - \frac{9}{x-4}$$

If we compare this result to the result we obtained in Example 6.54, we can see that they are identical. Whether we divided using polynomial division or synthetic division, we obtained the same result.

In general, synthetic division is the tool of choice for dividing polynomials by divisors of the form $x-c$. It is important to note that it works *only* for these kinds of divisors. Good old-fashioned polynomial long division for divisors of degree larger than 1 should be used in other cases. Also, take note that when a polynomial (of degree at least 1) is divided by $x-c$, the result is a polynomial of exactly one less degree.

6.6.6 Polynomial Division Homework

Divide.

1. $\dfrac{20x^4+x^3+2x^2}{4x^3}$

2. $\dfrac{20n^4+n^3+40n^2}{10n}$

3. $\dfrac{12x^4+24x^3+3x^2}{6x}$

4. $\dfrac{10n^4+50n^3+2n^2}{10n^2}$

5. $\dfrac{x^2-2x-71}{x+8}$

6. $\dfrac{n^2+13n+32}{n+5}$

7. $\dfrac{v^2-2v-89}{v-10}$

8. $\dfrac{a^2-4a-38}{a-8}$

9. $\dfrac{45p^2+56p+19}{9p+4}$

10. $\dfrac{10x^2-32x+9}{10x-2}$

11. $\dfrac{4r^2-r-1}{4r+3}$

12. $\dfrac{n^2-4}{n-2}$

13. $\dfrac{27b^2+87b+35}{3b+8}$

14. $\dfrac{4x^2-33x+28}{4x-5}$

15. $\dfrac{a^3+15a^2+49a-55}{a+7}$

16. $\dfrac{x^3-26x-41}{x+4}$

17. $\dfrac{3n^3+9n^2-64n-68}{n+6}$

18. $\dfrac{x^3-46x+22}{x+7}$

19. $\dfrac{9p^3+45p^2+27p-5}{9p+9}$

20. $\dfrac{r^3-r^2-16r+8}{r-4}$

21. $\dfrac{12n^3+12n^2-15n-4}{2n+3}$

22. $\dfrac{4v^3-21v^2+6v+19}{4v+3}$

23. $\dfrac{5x^4+45x^3+4x^2}{9x}$

24. $\dfrac{3k^3+4k^2+2k}{8k}$

25. $\dfrac{5p^4+16p^3+16p^2}{4p}$

26. $\dfrac{3m^4+18m^3+27m^2}{9m^2}$

27. $\dfrac{r^2-3r-53}{r-9}$

28. $\dfrac{b^2-10b+16}{b-7}$

29. $\dfrac{x^2+4x-26}{x+7}$

30. $\dfrac{x^2-10x+22}{x-4}$

31. $\dfrac{48k^2-70k+16}{6k-2}$

32. $\dfrac{n^2+7n+15}{n+4}$

33. $\dfrac{3m^2+9m-9}{3m-3}$

34. $\dfrac{2x^2-5x-8}{2x+3}$

35. $\dfrac{3v^2-32}{3v-9}$

36. $\dfrac{4n^2-23n-38}{4n+5}$

37. $\dfrac{8k^3-66k^2+12k+37}{k-8}$

38. $\dfrac{x^3-16x^2+71x-56}{x-8}$

39. $\dfrac{k^3-4k^2-6k+4}{k-1}$

40. $\dfrac{2n^3+21n^2+25n}{2n+3}$

41. $\dfrac{8m^3-57m^2+42}{8m+7}$

42. $\dfrac{2x^3+12x^2+4x-37}{2x+6}$

43. $\dfrac{24b^3-38b^2+29b-60}{4b-7}$

Perform the indicated operations given the set of functions.

44. Let $f(x)=x^3-2x^2-4x-5$ and $g(x)=x+2$, find $(f \div g)(x)$.

45. Let $f(n)=3n+5$ and $k(n)=n^2+5$, find $(k \div f)(n)$.

Use synthetic division to divide.

46. x^3+4x^2+4x+6 by $x+1$

47. $x^4+4x^3-28x^2+26x-17$ by $x-3$

48. x^3+3x^2-2x+5 by $x-1$

49. $x^4+5x^3+11x^2+13x-2$ by $x+2$

50. $x^3 + x^2 - 4x - 1$ by $x - 2$

51. $x^4 - 4x^3 - 6x^2 - 4x - 8$ by $x + 1$

52. $x^3 + 4x^2 - 6x - 5$ by $x - 2$

53. $x^4 + 10x^3 + 11x^2 - 15x - 14$ by $x + 2$

54. $x^3 + 11x^2 + 26x + 12$ by $x + 3$

55. $x^4 - 4x^3 - 3x^2 + 6x + 19$ by $x - 2$

6.7 Exponents & Polynomials: Answers to the Homework Exercises

Exponent Rules and Properties

1. 4^9

3. $12m^2n$

5. 3^{12}

7. $4u^6v^4$

9. 4^2

11. m^2

13. $4x^{10}y^{14}$

15. x^2y^6

17. $\dfrac{y^3}{512x^24}$

19. $2xy$

21. $x^4y^{16}z^4$

23. 4^7

25. $12x^3$

27. 4^{12}

29. x^3y^3

31. 3^4

33. $\dfrac{xy^3}{4}$

35. $8u^{18}v^6$

37. $\dfrac{4a^2}{3}$

39. $\dfrac{x^2y^5}{2}$

41. $2y^2$

43. $256q^4r^8$

45. b^11

47. $\dfrac{x^5y^{16}}{32}$

49. $\dfrac{1}{x^3y^2}$

51. $\dfrac{u^2}{12v^5}$

53. $16a^{12}b^{12}$

55. $16m^4n^6$

57. $\dfrac{u}{2v}$

59. $2x^4y^5$

61. $\dfrac{1}{h^3j^6k}$

63. $\dfrac{x^2}{y^4z^4}$

65. $2x^3y^2$

67. $\dfrac{32}{m^5n^{15}}$

69. $\dfrac{x^5y^8}{4}$

71. $\dfrac{y}{2x^4}$

73. $\dfrac{x^4y^8}{4}$

75. $\dfrac{2x}{y^3}$

77. $4y^5$

79. $\dfrac{a^3}{2b^3}$

81. $\dfrac{x^{30}z^6}{16y^4}$

83. $\dfrac{mn^7}{p^5}$

Scientific Notation

1. 8.85×10^2

3. 3.9×10^{-2}

5. 1.09×10^0

7. 870,000

9. 2

11. 50,000

13. 1.4×10^{-3}

15. 1.56×10^{-3}

17. 5.541×10^{-5}

19. 2.887×10^{-6}

21. 1.196×10^{-2}

23. 1.715×10^{14}

25. 4.6×10^2

27. 1.034×10^6

29. 5.018×10^6

31. 9.836×10^{-1}

33. 1.177×10^{-16}

35. 2.91×10^{-2}

37. 2.52×10^3

39. 3.939×10^9

41. 1.372×10^3

Add and Subtract Polynomial Expressions

1. 3

3. -10

5. -7

7. 5

9. 12

11. $3p^4 - 3p$

13. $-n^3 + 10n^2$

15. $5n^4 + 5n$

17. $13p^3$

19. $3n^3 + 8$

21. $2b^4 + 2b + 10$

23. $-5x^4 + 14x^3 - 1$

25. $7a^4 - 3a^2 - 2a$

27. $p^2 + 4p - 6$

29. $5b^3 + 12b^2 + 5$

31. $n^3 - 5n^2 + 3$

33. $-12n^4 + n^2 + 7$

35. $r^4 - 3r^3 + 7r^2 + 1$

37. $9n^4 + 2n^3 + 6n^2$

39. $-3b^4 + 13b^3 - 7b^2 - 11b + 19$

41. $2x^4 - x^3 - 4x + 2$

Multiply Polynomial Expressions

1. $6p - 42$

3. $20m^5 + 20m^4$

5. $56b^2 - 19b - 15$

7. $15v^2 - 26v + 8$

9. $30x^2 - 14xy - 4y^2$

11. $56x^2 + 61xy + 15y^2$

13. $12n^3 - 20n^2 + 38n - 20$

15. $48n^4 - 16n^3 + 64n^2 - 6n + 36$

17. $18x^2 - 15x - 12$

19. $7x^2 - 49x + 70$

21. $32k^2 + 16k$

23. $12r - 21$

25. $4r^2 + 40r + 64$

27. $6a^2 - 44a - 32$

29. $16u^2 + 10uv - 21v^2$

31. $5a^2 - 7ab - 24b^2$

33. $8b^3 - 4b^2 - 4b - 12$

35. $14a^4 + 30a^3 - 13a^2 - 12a + 3$

37. $10x^2 - 55x + 60$

39. $40x^2 - 10x - 5$

41. $4x^3 + 25x^2 + 25x$

43. $-2n^3 - 15n^2 - 25n$

Special Products

1. $x^2 - 64$

3. $1 - 49n^2$

5. $16x^2 - 64$

7. $16m^2 - 64n^2$

9. $a^2 + 10a + 25$

11. $p^2 + 14p + 49$

13. $25m^2 - 80m + 64$

15. $4x^2 + 8xy + 4y^2$

17. $4 + 20x + 25x^2$

19. $n^2 - 25$

21. $a^2 - 16$

23. $64m^2 - 25$

25. $b^2 - 49$

27. $9y^2 - 9x^2$

29. $v^2 + 8v + 16$

31. $49k^2 - 98k + 49$

33. $9a^2 + 18ab + 9b^2$

35. $64x^2 + 80xy + 25y^2$

37. $64n^2 - 49$

39. $49x^2 + 98x + 49$

Polynomial Division

1. $5x + \frac{1}{4} + \frac{1}{2x}$

3. $2x^3 + 4x^2 + \frac{x}{2}$

5. $x - 10 + \frac{9}{x+8}$

7. $v + 8 - \frac{9}{v-10}$

9. $5p + 4 + \frac{3}{9p+4}$

11. $r - 1 + \frac{2}{4r+3}$

13. $9b + 5 - \frac{5}{3b+8}$

15. $a^2 + 8a - 7 - \frac{6}{a+7}$

17. $3n^2 - 9n - 10 - \frac{8}{n+6}$

19. $p^2 + 4p - 1 + \frac{4}{9p+9}$

21. $6n^2 - 3n - 3 + \dfrac{5}{2n+3}$

23. $\dfrac{5x^3}{9} + 5x^2 + \dfrac{4x}{9}$

25. $\dfrac{5p^3}{4} + 4p^2 + 4p$

27. $r + 6 + \dfrac{1}{r-9}$

29. $x - 3 - \dfrac{5}{x+7}$

31. $8k - 9 - \dfrac{1}{3k-1}$

33. $m + 4 + \dfrac{1}{m-1}$

35. $v + 3 - \dfrac{5}{3v-9}$

37. $8k^2 - 2k - 4 + \dfrac{5}{k-8}$

39. $k^2 - 3k - 9 - \dfrac{5}{k-1}$

41. $m^2 - 8m + 7 - \dfrac{7}{8m+7}$

43. $6b^2 + b + 9 + \dfrac{3}{4b-7}$

45. $\dfrac{1}{3}n - \dfrac{5}{9} + \dfrac{\frac{70}{9}}{3n+5}$

47. $x^3 + 7x^2 - 7x + 5 - \dfrac{2}{x-3}$

49. $x^3 + 3x^2 + 5x + 3 - \dfrac{8}{x+2}$

51. $x^3 - 5x^2 - x - 3 - \dfrac{5}{x+1}$

53. $x^3 + 8x^2 - 5x - 5 - \dfrac{4}{x+2}$

55. $x^3 - 2x^2 - 7x - 8 + \dfrac{3}{x-2}$

Chapter 7

Factoring Expressions and Solving by Factoring

Chapter Objectives

By the end of this chapter, the student should be able to

- Factor a greatest common factor
- Factor by grouping including rearranging terms
- Factor by applying special-product formulas
- Factor trinomials by using a general strategy including by substitution
- Solve equations and applications by factoring

After multiplying polynomials, we discuss rewriting polynomials in its factored form. There are many applications to factoring polynomials in algebra. The deeper into algebra, the factoring techniques we discuss in this chapter become more valuable. We use the factored form of polynomials to help solve equations, see behaviors of graphs, work with rational expressions, and more. In the next chapter, simplifying is solely dependent on factoring and the more efficient we are at factoring, the more efficient we will be at simplifying rational expressions.

7.1 Greatest common factor and grouping

In this lesson, we focus on factoring using the greatest common factor, GCF, of a polynomial. When we multiplied polynomials, we multiplied monomials by polynomials by distributing, such as

$$4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x^2$$

We work out the same problem, but backwards. We will start with $8x^2 - 12x^3 + 32x^2$ and obtain its factored form.

First, we have to identify the GCF of a polynomial. We introduce the GCF of a polynomial by looking at an example in arithmetic. The method in which we obtained the GCF between numbers in arithmetic is the same method we use to obtain the GCF with polynomials.

7.1.1 Finding the greatest common factor

Definition

The *factored form* of a number or expression is the expression written as a product of factors.

The ***greatest common factor*** (GCF) of a polynomial is the largest polynomial that is a factor of all terms in the polynomial.

Example 7.1

Find the GCF of 15, 24, and 27.

Solution.

First we obtain the prime factorization of each number:

$$\begin{aligned} 15 &= 3 \cdot 5 \\ 24 &= 2^3 \cdot 3 \\ 27 &= 3^3 \end{aligned}$$

Next, we take only the common factors and if any common factors repeat, we take the factor with the smallest exponent. Recall, the GCF is the largest factor that divides into all numbers. We need to take 3 (we only take 3^1 because there is only one three in common in all three numbers). Notice, there are no other factors in common with all three numbers. Hence, $\text{GCF}(15, 24, 27) = 3$.

When finding the GCF of a polynomial, we use the same method and the only difference is that we have variables in the expression.

Example 7.2

Find the GCF of $24x^4y^2z, 18x^2y^4$, and $12x^3yz^5$

Solution.

First we obtain the prime factorization of each monomial:

$$\begin{aligned} 24x^4y^2z &= 2^3 \cdot 3 \cdot x^4 \cdot y^2 \cdot z \\ 18x^2y^4 &= 2 \cdot 3^2 \cdot x^2 \cdot y^4 \\ 12x^3yz^5 &= 2^2 \cdot 3 \cdot x^3 \cdot y \cdot z^5 \end{aligned}$$

Next, we take only the common factors and if any common factors repeat, we take the factor with the smallest exponent. Recall, the GCF is the largest factor that divides into all terms in the expression. We need to take $2 \cdot 3 \cdot x^2 \cdot y \cdot z$. Hence, $\text{GCF}(24x^4y^2z, 18x^2y^4, 12x^3yz^5) = 6x^2yz$.

7.1.2 Factoring the greatest common factor

Once we obtain the GCF, we can start factoring the GCF from an expression. Eventually, we want to be able to see the GCF quickly and begin factoring right away. Of course, only with practice we obtain better skills. Let's try an example.

Example 7.3

Factor out the GCF: $4x^2 - 20x + 16$

Solution.

Looking at each term, let's write the prime factorization of each term:

$$4x^2 = 2^2 \cdot x^2$$
$$20x = 2^2 \cdot 5 \cdot x$$
$$16 = 2^4$$

We need to take 2^2. Hence, GCF$(4x^2, 20x, 16) = 2^2 = 4$. Let's rewrite each term in the expression as the product of the GCF and the factors left:

$4x^2 - 20x + 16$	Rewrite with the GCF 4
$4 \cdot x^2 - 4 \cdot 5x + 4 \cdot 4$	Rewrite the expression with the GCF and parenthesis
$4(x^2 - 5x + 4)$	Factored form

Steps for factoring out the greatest common factor

Step 1. Find the GCF of the expression.

Step 2. Rewrite each term as a product of the GCF and the remaining factors.

Step 3. Rewrite as a product of the GCF and the remaining factors in parenthesis.

Step 4. ✓Verify the factored form by multiplying. The product should be the original expression.

Example 7.4

Factor out the GCF: $25x^4 - 15x^3 + 20x^2$

Solution.

Looking at each term, let's start by writing the prime factorization of each term.

$$25x^4 = 5^2 \cdot x^4$$
$$15x^3 = 3 \cdot 5 \cdot x^3$$
$$20x^2 = 2^2 \cdot 5 \cdot x^2$$

Step 1. We need to take $5x^2$. Hence, GCF$(25x^4, 15x^3, 20x^2) = 5x^2$.

Step 2. Let's rewrite each term in the expression as the product of the GCF and the factors left:

$25x^4 - 15x^3 + 20x^2$	Rewrite with the GCF $5x^2$
$5x^2 \cdot 5x^2 - 5x^2 \cdot 3x + 5x^2 \cdot 4$	

Step 3. Rewrite the expression with the GCF and the remaining factors in parenthesis:

$5x^2 \cdot 5x^2 - 5x^2 \cdot 3x + 5x^2 \cdot 4$	Rewrite the expression with the GCF and parenthesis
$5x^2(5x^2 - 3x + 4)$	Factored form

Step 4. Let's verify the factored form:

$5x^2(5x^2 - 3x + 4)$	Distribute the GCF
$25x^4 - 15x^3 + 20x^2$	✓ Original expression

Thus, the factored form is $5x^2(5x^2 - 3x + 4)$.

Example 7.5

Factor out the GCF: $3x^3y^2z + 5x^4y^3z^5 - 4xy^4$

Solution.

Looking at each term, let's start by writing the prime factorization of each term.

$$3x^3y^2z = 3 \cdot x^3 \cdot y^2 \cdot z$$
$$5x^4y^3z^5 = 5 \cdot x^4 \cdot y^3 \cdot z^5$$
$$4xy^4 = 2^2 \cdot x \cdot y^4$$

Step 1. We need to take xy^2. Hence, $\text{GCF}(3x^3y^2z, 5x^4y^3z^5, 4xy^4) = xy^2$.

Step 2. Let's rewrite each term in the expression as the product of the GCF and the factors left:

$$3x^3y^2z + 5x^4y^3z^5 - 4xy^4 \qquad \text{Rewrite with the GCF } xy^2$$
$$xy^2 \cdot 3x^2z + xy^2 \cdot 5x^3yz^5 - xy^2 \cdot 4y^2$$

Step 3. Rewrite the expression with the GCF and the remaining factors in parenthesis:

$$xy^2 \cdot 3x^2z + xy^2 \cdot 5x^3yz^5 - xy^2 \cdot 4y^2 \qquad \text{Rewrite the expression with the GCF and parenthesis}$$
$$xy^2(3x^2z + 5x^2yz^5 - 4y^2) \qquad \text{Factored form}$$

Step 4. Let's verify the factored form:

$$xy^2(3x^2z + 5x^2yz^5 - 4y^2) \qquad \text{Distribute the GCF}$$
$$3x^3y^2z + 5x^4y^3z^5 - 4xy^4 \qquad \checkmark \text{ Original expression}$$

Thus, the factored form is $xy^2(3x^2z + 5x^2yz^5 - 4y^2)$.

World Note

The first recorded algorithm for finding the greatest common factor comes from Greek mathematician *Euclid* around the year 300 BC.

Let's try an example with eye-balling the GCF and rewriting in factored form.

Example 7.6

Factor out the GCF: $21x^3 + 14x^2 + 7x$

Solution.

Looking at the coefficients, we can see that there is common factor of 7 in each term. Furthermore, we see a factor of x in common in all three terms. Hence, we take $7x$ as the GCF. Notice we didn't take a larger exponent on x because only one factor of x is common in all three terms.

Let's rewrite the expression in factored form.

$$\begin{aligned} 21x^3 + 14x^2 + 7x & \quad \text{Rewrite with the GCF } 7x \\ 7x \cdot 3x^2 + 7x \cdot 2x + 7x \cdot 1 & \quad \text{Rewrite the expression with the GCF and parenthesis} \\ 7x(3x^2 + 2x + 1) & \quad \text{Factored form} \end{aligned}$$

We can always verify the factored form by distributing the $7x$ and obtaining the original expression.

7.1.3 A binomial as the greatest common factor

As part of a general strategy for factoring, we always look for a GCF. Sometimes the GCF is a monomial, like in the previous examples, or a binomial. Here we discuss factoring a polynomial where the GCF is a binomial. We usually call this *factor by grouping.* Consider the below example.

Example 7.7

Factor: $3ax - 7bx$

Solution.

$$\begin{aligned} 3ax - 7bx & \quad \text{Both have } x \text{ in common, factor it out} \\ x(3a - 7b) & \quad \text{Factored form} \end{aligned}$$

Let's make this interesting. Let's apply this same method, but instead of the GCF being x, let the GCF be the binomial $2a + 5b$.

Example 7.8

Factor: $3a(2a + 5b) - 7b(2a + 5b)$

Solution.

$$\begin{aligned} 3a(2a + 5b) - 7b(2a + 5b) & \quad \text{Both have } (2a + 5b) \text{ in common, factor it out} \\ (2a + 5b)(3a - 7b) & \quad \text{Factored form} \end{aligned}$$

7.1.4 Factor by grouping

In example 7.8, we factored out a GCF of $(2a + 5b)$ the same way we factored out an x in example 7.7. This process can be extended to factor expressions where there isn't a GCF . We will use a process known as *factor by grouping.* Factor by grouping is a method used to factor polynomials when there is at least four terms in the expression. Take the next example.

Example 7.9

Multiply: $(2a + 3)(5b + 2)$

Solution.

$$\begin{aligned} (2a + 3)(5b + 2) & \quad \text{Distribute } (2a + 3) \text{ into second parenthesis} \\ 5b(2a + 3) + 2(2a + 3) & \quad \text{Distribute} \\ 10ab + 15b + 4a + 6 & \quad \text{Product} \end{aligned}$$

Notice the product has four terms none of which share a common factor.

Steps for factoring by grouping

To factor by grouping, we first notice the polynomial expression obtains four terms.

Step 1. Group two sets of two terms, e.g., $ax + ay + bx + by = (ax + ay) + (bx + by)$.

Step 2. Factor the GCF from each group, e.g., $a(x + y) + b(x + y)$

Step 3. Factor the GCF from the expression , e.g., $(x + y)(a + b)$.

Example 7.10

Factor: $10ab + 15b + 4a + 6$

Solution.

Notice we have 4 terms none of which share a common factor. Hence, we use factor by grouping.

Step 1. Group two sets of two terms:

$$\begin{array}{rl} 10ab + 15b + 4a + 6 & \text{Group the first two terms and the last two terms} \\ (10ab + 15b) + (4a + 6) & \end{array}$$

Step 2. Factor the GCF from each group:

$$\begin{array}{rl} (10ab + 15b) + (4a + 6) & \text{Factor } 5b \text{ from the first group and 2 from the second group} \\ 5b(2a + 3) + 2(2a + 3) & \end{array}$$

Step 3. Factor the GCF from the expression:

$$\begin{array}{rl} 5b(2a + 3) + 2(2a + 3) & \text{Factor the GCF } (2a + 3) \\ (2a + 3)(5b + 2) & \text{Factored form} \end{array}$$

Recall, we can verify the factored form by multiplying the binomials and obtaining the original expression.

Example 7.11

Factor: $6x^2 + 9xy - 14x - 21y$

Solution.

Notice we have 4 terms none of which share a common factor. Hence, we use factor by grouping.

Step 1. Group two sets of two terms:

$$\begin{array}{rl} 6x^2 + 9xy - 14x - 21y & \text{Group the first two terms and the last two terms} \\ (6x^2 + 9xy) + (-14x - 21y) & \end{array}$$

Step 2. Factor the GCF from each group:

$$(6x^2 + 9xy) + (-14x - 21y) \qquad \text{Factor } 3x \text{ from the first group and } -7 \text{ from the second group}$$
$$3x(2x + 3y) - 7(2x + 3y)$$

Step 3. Factor the GCF from the expression:

$$3x(2x + 3y) - 7(2x + 3y) \qquad \text{Factor the GCF } (2x + 3y)$$
$$(2x + 3y)(3x - 7) \qquad \text{Factored form}$$

Recall, we can verify the factored form by multiplying the binomials and obtaining the original expression.

Note

Notice after **Step 2.**, we want the binomial's GCF to be identical so that we can factor it out in **Step 3.** Be sure these binomials are identical. A common error is when the binomials aren't identical (sometimes by a negative) and students factor anyways.

Example 7.12

Factor: $5xy - 8x - 10y + 16$

Solution.

Notice we have 4 terms none of which share a common factor. Hence, we use factor by grouping.

Step 1. Group two sets of two terms:

$$5xy - 8x - 10y + 16 \qquad \text{Group the first two terms and the last two terms}$$
$$(5xy - 8x) + (-10y + 16)$$

Step 2. Factor the GCF from each group:

$$(5xy - 8x) + (-10y + 16) \qquad \text{Factor } x \text{ from the first group and } -2 \text{ from the second group}$$
$$x(5y - 8) - 2(5y - 8) \qquad \text{Both binomials are identical}$$

Step 3. Factor the GCF from the expression:

$$x(5y - 8) - 2(5y - 8) \qquad \text{Factor the GCF } (5y - 8)$$
$$(5y - 8)(x - 2) \qquad \text{Factored form}$$

Recall, we can verify the factored form by multiplying the binomials and obtaining the original expression.

Example 7.13

Factor: $12ab - 14a - 6b + 7$

Solution.

Notice we have 4 terms none of which share a common factor. Hence, we use factor by grouping.

Step 1. Group two sets of two terms:

$$\begin{aligned} 12ab - 14a - 6b + 7 &\quad \text{Group the first two terms and the last two terms} \\ (12ab - 14a) + (-6b + 7) & \end{aligned}$$

Step 2. Factor the GCF from each group:

$$\begin{aligned} (12ab - 14a) + (-6b + 7) &\quad \text{Factor } 2a \text{ from the first group and } -1 \\ &\quad \text{from the second group} \\ 2a(6b - 7) - 1(6b - 7) &\quad \text{Both binomials are identical} \end{aligned}$$

Step 3. Factor the GCF from the expression:

$$\begin{aligned} 2a(6b - 7) - 1(6b - 7) &\quad \text{Factor the GCF } (6b - 7) \\ (6b - 7)(2a - 1) &\quad \text{Factored form} \end{aligned}$$

Careful in these types of expressions, where we factor the entire second binomial and are left with the term 1. This occurs sometimes with factoring and it's important to always write the 1 in **Step 2.** so that we do not forget it is there. Recall, we can verify the factored form by multiplying the binomials and obtaining the original expression.

7.1.5 Factor by grouping by rearranging terms

Sometimes after completing **Step 2.**, the binomials aren't identical (by more than a negative sign). At this point we must return to the original problem and rearrange the terms so that when we factor by grouping, we obtain identical binomials in **Step 2.**

Example 7.14

Factor: $4a^2 - 21b^3 + 6ab - 14ab^2$

Solution.

Notice we have 4 terms none of which share a common factor. Hence, we use factor by grouping.

Step 1. Group two sets of two terms:

$$\begin{aligned} 4a^2 - 21b^3 + 6ab - 14ab^2 &\quad \text{Group the first two terms and the last two terms} \\ (4a^2 - 21b^3) + (6ab - 14ab^2) & \end{aligned}$$

Step 2. Factor the GCF from each group:

$$\begin{aligned} (4a^2 - 21b^3) + (6ab - 14ab^2) &\quad \text{Factor } 2ab \text{ from the second group} \\ (4a^2 - 21b^3) + 2a(3b - 7b^2) &\quad \text{Binomials are NOT identical} \end{aligned}$$

Since these binomials aren't identical, we return to the original expression and rearrange the terms. Let's try moving $6ab$ to the first group and $-21b^3$ to the second group.

Step 1. Group two sets of two terms:

$$\begin{array}{rl} 4a^2 + 6ab - 21b^3 - 14ab^2 & \text{Group the first two terms and the last two terms} \\ (4a^2 + 6ab) + (-21b^3 - 14ab^2) & \end{array}$$

Step 2. Factor the GCF from each group:

$$\begin{array}{rl} (4a^2 + 6ab) + (-21b^3 - 14ab^2) & \text{Factor } 2a \text{ from the first group and } -7b^2 \\ & \text{from the second group} \\ 2a(2a + 3b) - 7b^2(3b + 2a) & \text{Rewrite so the binomials are identical} \\ 2a(2a + 3b) - 7b^2(2a + 3b) & \text{Binomials are identical} \end{array}$$

Step 3. Factor the GCF from the expression:

$$\begin{array}{rl} 2a(2a + 3b) - 7b^2(2a + 3b) & \text{Factor the GCF } (2a + 3b) \\ (2a + 3b)(2a - 7b^2) & \text{Factored form} \end{array}$$

Recall, we can verify the factored form by multiplying the binomials and obtaining the original expression.

Example 7.15

Factor: $8xy - 12y + 15 - 10x$

Solution.

Notice we have 4 terms none of which share a common factor. Hence, we use factor by grouping.

Step 1. Group two sets of two terms:

$$\begin{array}{rl} 8xy - 12y + 15 - 10x & \text{Group the first two terms and the last two terms} \\ (8xy - 12y) + (15 - 10x) & \end{array}$$

Step 2. Factor the GCF from each group:

$$\begin{array}{rl} (8xy - 12y) + (15 - 10x) & \text{Factor } 4y \text{ from the first group and } 5 \\ & \text{from the second group} \\ 4y(2x - 3) + 5(3 - 2x) & \text{Binomials are NOT identical, but VERY close} \end{array}$$

Since these binomials aren't identical but close to it, we can think about it some more. These binomials would be identical if only the 3 and $-2x$ in the second binomial were switched. Let's factor a -1 out of the second binomial:

$$\begin{array}{rl} 4y(2x - 3) + 5(3 - 2x) & \text{Factor a } -1 \text{ from the second binomial} \\ 4y(2x - 3) + 5 \cdot -1(-3 + 2x) & \text{Rewrite so the binomials are identical} \\ 4y(2x - 3) - 5(2x - 3) & \text{Binomials are identical} \end{array}$$

Step 3. Factor the GCF from the expression:

$$\begin{array}{rl} 4y(2x - 3) - 5(2x - 3) & \text{Factor the GCF } (2x - 3) \\ (2x - 3)(4y - 5) & \text{Factored form} \end{array}$$

Recall, we can verify the factored form by multiplying the binomials and obtaining the original expression.

Note

If the binomials are $(a-b)$ and $(b-a)$, we will factor -1 out of one of the binomials to obtain identical binomials.

$(b-a)$	Factor out -1
$-1(-b+a)$	Apply the commutative property to the addition
$-1(a-b)$	Now the binomial is written as subtraction

World Note

Sofia Kovalevskaya of Russia was the first woman on the editorial staff of a mathematical journal in the late 19^{th} century. She also researched the planet Saturn's rotating rings.

7.1.6 Greatest Common Factor and Grouping Homework

Factor the greatest common factor.

1. $9 + 8b^2$

2. $45x^2 - 25$

3. $56 - 35p$

4. $7ab - 35a^2b$

5. $-3a^2b + 6a^3b^2$

6. $-5x^2 - 5x^3 - 15x^4$

7. $20x^4 - 30x + 30$

8. $28m^4 + 40m^3 + 8$

9. $30b^9 + 5ab - 15a^2$

10. $-48a^2b^2 - 56a^3b - 56a^5b$

11. $20x^8y^2z^2 + 15x^5y^2z + 35x^3y^3z$

12. $50x^2y + 10y^2 + 70xz^2$

13. $30qpr - 5qp + 5q$

14. $-18n^5 + 3n^3 - 21n + 3$

15. $-40x^{11} - 20x^{12} + 50x^{13} - 50x^{14}$

16. $-32mn^8 + 4m^6n + 12mn^4 + 16mn$

17. $x - 5$

18. $1 + 2n^2$

19. $50x - 80y$

20. $27x^2y^5 - 72x^3y^2$

21. $8x^3y^2 + 4x^3$

22. $-32n^9 + 32n^6 + 40n^5$

23. $21p^6 + 30p^2 + 27$

24. $-10x^4 + 20x^2 + 12x$

25. $27y^7 + 12y^2x + 9y^2$

26. $30m^6 + 15mn^2 - 25$

27. $3p + 12q - 15q^2r^2$

28. $30y^4z^3x^5 + 50y^4z^5 - 10y^4z^3x$

29. $28b + 14b^2 + 35b^3 + 7b^5$

30. $30a^8 + 6a^5 + 27a^3 + 21a^2$

31. $-24x^6 - 4x^4 + 12x^3 + 4x^2$

32. $-10y^7 + 6y^{10} - 4y^{10}x - 8y^8x$

Factor each completely.

33. $40r^3 - 8r^2 - 25r + 5$

34. $3n^3 - 2n^2 - 9n + 6$

35. $15b^3 + 21b^2 - 35b - 49$

36. $3x^3 + 15x^2 + 2x + 10$

37. $35x^3 - 28x^2 - 20x + 16$

38. $7xy - 49x + 5y - 35$

39. $32xy + 40x^2 + 12y + 15x$

40. $16xy - 56x + 2y - 7$

41. $2xy - 8x^2 + 7y^3 - 28y^2x$

42. $40xy + 35x - 8y^2 - 7y$

43. $32uv - 20u + 24v - 15$

44. $10xy + 30 + 25x + 12y$

45. $3uv + 14u - 6u^2 - 7v$

46. $16xy - 3x - 6x^2 + 8y$

47. $35x^3 - 10x^2 - 56x + 16$

48. $14v^3 + 10v^2 - 7v - 5$

49. $6x^3 - 48x^2 + 5x - 40$

50. $28p^3 + 21p^2 + 20p + 15$

51. $7n^3 + 21n^2 - 5n - 15$

52. $42r^3 - 49r^2 + 18r - 21$

53. $15ab - 6a + 5b^3 - 2b^2$

54. $3mn - 8m + 15n - 40$

55. $5mn + 2m - 25n - 10$

56. $8xy + 56x - y - 7$

57. $4uv + 14u^2 + 12v + 42u$

58. $24xy + 25y^2 - 20x - 30y^3$

59. $56ab + 14 - 49a - 16b$

7.2 Factoring trinomials of the form $x^2 + bx + c$

Factoring with three terms, or trinomials, is the most important technique, especially in further algebra. Since factoring is a product of factors, we first look at multiplying to develop the process of factoring trinomials.

7.2.1 Factoring trinomials of the form $x^2 + bx + c$

If we multiply $(x + p)(x + q)$, we would get

$$x^2 + px + qx + pq$$

$$x^2 + (p + q)x + pq$$

Notice the two factors of the last coefficient must add up to be the middle coefficient, i.e.,

$$p \cdot q = c \text{ and } p + q = b$$

Hence, if we can find two numbers whose sum is b and that multiply to c, then we can split the middle term and factor by grouping.

Steps for factoring trinomials of the form $x^2 + bx + c$

Step 1. Find two numbers, p and q, whose sum is b and product is c.

Step 2. Rewrite the expression so that the middle term is split into two terms, p and q.

Step 3. Factor by grouping.

Step 4. Verify the factored form by finding the product.

Example 7.16

Factor: $x^2 + 9x + 18$

Solution.

First we identify $b = 9$ and $c = 18$. We ask ourselves,“ What two numbers multiply to 18 that add up to 9?”

Step 1. Find two numbers whose sum is 9 and product is 18:

p and q	Product	Sum
2,9	18	11
3,6	18	9
1,18	18	19

We can see from the table that 3 and 6 are the two numbers whose product is 18 and sum is 9. We use these two numbers in **Step 2.**

Step 2. Rewrite the expression so that the middle term is split into two terms, $3x$ and $6x$:

$$x^2 + 9x + 18$$

$$x^2 \underbrace{+3x + 6x}_{\text{sum is } 9x} + 18$$

Step 3. Factor by grouping.

$x^2 + 6x + 3x + 18$	Group the first two terms and the last two terms
$(x^2 + 3x) + (6x + 18)$	Factor x from the first group and 6 from the second group
$x(x + 3) + 6(x + 3)$	Factor the GCF $(x + 3)$
$(x + 3)(x + 6)$	Factored form

Step 4. Verify the factored form by finding the product:

$(x + 3)(x + 6)$	FOIL
$x^2 + 6x + 3x + 18$	Combine like terms
$x^2 + 9x + 18$	✓Original expression

Thus, the factored form is $(x + 3)(x + 6)$.

Example 7.17

Factor: $x^2 - 4x + 3$

Solution.

First we identify $b = -4$ and $c = 3$. We ask ourselves,“ What two numbers multiply to 3 that add up to -4?”

Step 1. Find two numbers whose sum is -4 and product is 3:

p and q	Product	Sum
1,3	3	4
$-1, -3$	3	-4

We can see from the table that -1 and -3 are the two numbers whose product is 3 and sum is -4. We use these two numbers in **Step 2.**

Step 2. Rewrite the expression so that the middle term is split into two terms, $-1x$ and $-3x$:

$$x^2 - 4x + 3$$

$$x^2 \underbrace{-1x - 3x}_{\text{sum is } -4x} + 3$$

Step 3. Factor by grouping.

$x^2 - 1x - 3x + 3$	Group the first two terms and the last two terms
$(x^2 - 1x) + (-3x + 3)$	Factor x from the first group and -3 from the second group
$x(x - 1) - 3(x - 1)$	Factor the GCF $(x - 1)$
$(x - 1)(x - 3)$	Factored form

Step 4. Verify the factored form by finding the product:

$$\begin{aligned} (x-1)(x-3) &\quad \text{FOIL} \\ x^2-1x-3x+3 &\quad \text{Combine like terms} \\ x^2-4x+3 &\quad \checkmark\text{Original expression} \end{aligned}$$

Thus, the factored form is $(x-1)(x-3)$.

Example 7.18

Factor: $x^2-8x-20$

Solution.

First we identify $b=-8$ and $c=-20$. We ask ourselves,“ What two numbers multiply to -20 that add up to -8?”

Step 1. Find two numbers whose sum is -8 and product is -20:

p and q	Product	Sum
$-4, 5$	-20	1
$4, -5$	-20	-1
$-2, 10$	-20	8
$2, -10$	-20	-8
$1, -20$	-20	-19
$-1, 20$	-20	19

We can see from the table that 2 and -10 are the two numbers whose product is -20 and sum is -8. We use these two numbers in **Step 2.**

Step 2. Rewrite the expression so that the middle term is split into two terms, $2x$ and $-10x$:

$$x^2-8x-20$$

$$x^2\underbrace{+2x-10x}_{\text{sum is } -8x}-20$$

Step 3. Factor by grouping.

$$\begin{aligned} x^2+2x-10x-20 &\quad \text{Group the first two terms and the last two terms} \\ (x^2+2x)+(-10x-20) &\quad \text{Factor } x \text{ from the first group and } -10 \\ &\quad \text{from the second group} \\ x(x+2)-10(x+2) &\quad \text{Factor the GCF } (x+2) \\ (x+2)(x-10) &\quad \text{Factored form} \end{aligned}$$

Step 4. Verify the factored form by finding the product:

$$\begin{aligned} (x+2)(x-10) &\quad \text{FOIL} \\ x^2+2x-10x-20 &\quad \text{Combine like terms} \\ x^2-8x-20 &\quad \checkmark\text{Original expression} \end{aligned}$$

Thus, the factored form is $(x+2)(x-10)$.

Example 7.19

Factor: $a^2 - 9ab + 14b^2$

Solution.

First we identify $b = -9$ and $c = 14$. We ask ourselves," What two numbers multiply to 14 that add up to -9?"

Step 1. Find two numbers whose sum is -9 and product is 14:

p and q	Product	Sum
$2, 7$	14	9
$-2, -7$	14	-9
$-1, -14$	14	-15
$1, 14$	14	15

We can see from the table that -2 and -7 are the two numbers whose product is 14 and sum is -9. We use these two numbers in **Step 2.**

Step 2. Rewrite the expression so that the middle term is split into two terms, $-2ab$ and $-7ab$:

$$a^2 - 9ab + 14b^2$$

$$a^2 \underbrace{-2ab - 7ab}_{\text{sum is } -9ab} + 14b^2$$

Step 3. Factor by grouping.

$$\begin{aligned}
a^2 - 2ab - 7ab + 14b^2 &\quad \text{Group the first two terms and the last two terms}\\
(a^2 - 2ab) + (-7ab + 14b^2) &\quad \text{Factor } a \text{ from the first group and } -7b\\
&\quad \text{from the second group}\\
a(a - 2b) - 7b(a - 2b) &\quad \text{Factor the GCF } (a - 2b)\\
(a - 2b)(a - 7b) &\quad \text{Factored form}
\end{aligned}$$

Step 4. Verify the factored form by finding the product:

$$\begin{aligned}
(a - 2b)(a - 7b) &\quad \text{FOIL}\\
a^2 - 2ab - 7ab + 14b^2 &\quad \text{Combine like terms}\\
a^2 - 9ab + 14b^2 &\quad \checkmark\text{Original expression}
\end{aligned}$$

Thus, the factored form is $(a - 2b)(a - 7b)$.

Note

There is a shortcut for factoring expressions of the type $x^2 + bx + c$. Once we identify the two numbers, p and q, whose product is c and sum is b, we can see these two numbers are the numbers in the factored form, i.e., $(x + p)(x + q)$. We can use this shortcut only when the coefficient of x^2 is 1. (We discuss when the coefficient is a number other than 1 in the next section.)

Example 7.20

Factor: $x^2 - 7x - 18$

Solution.

First we identify $b = -7$ and $c = -18$. We ask ourselves," What two numbers multiply to -18 that add up to -7?"

p and q	Product	Sum
$-2, 9$	-18	7
$2, -9$	-18	-7
$-1, 18$	-18	17
$1, -18$	-18	-17

We can see from the table that 2 and -9 are the two numbers whose product is -18 and sum is -7. We use these two numbers to rewrite the expression in factored form:

$$x^2 - 7x - 18$$

$$(x+2)(x-9)$$

We can always verify the factored form by multiplying and obtaining the original expression.

Example 7.21

Factor: $m^2 - mn - 30n^2$

Solution.

First we identify $b = -1$ and $c = -30$. We ask ourselves," What two numbers multiply to -30 that add up to -1?"

p and q	Product	Sum
$-2, 15$	-30	13
$2, -15$	-30	-13
$5, -6$	-30	-1
$-5, 6$	-30	1
$1, -30$	-30	-29
$-1, 30$	-30	29

We can see from the table that 5 and -6 are the two numbers whose product is -30 and sum is -1. We use these two numbers to rewrite the expression in factored form:

$$m^2 - mn - 30n^2$$

$$(m+5n)(m-6n)$$

We can always verify the factored form by multiplying and obtaining the original expression.

Example 7.22

Factor: $x^2 + 2x + 6$

Solution.

First we identify $b = 2$ and $c = 6$. We ask ourselves," What two numbers multiply to 6 that add up to 2?"

p and q	Product	Sum
$2, 3$	6	5
$-2, -3$	6	-5
$1, 6$	6	7
$-1, -6$	6	-7

We can see from the table that there aren't any factors of 6 whose sum is 2. We only obtain sums with 5 and 7's. In this case, we call this trinomial *not factorable*, or better yet, the trinomial is *prime*.

Note

If a trinomial (or polynomial) is not factorable, then we say we the trinomial is *prime*.

7.2.2 Factoring trinomials of the form $x^2 + bx + c$ with a greatest common factor

Factoring the GCF is always the first step in factoring expressions. If all terms have a common factor, we, first, factor the GCF and then factor as usual.

Example 7.23

Factor: $3x^2 - 24x + 45$

Solution.

Notice all three terms have a common factor of 3. We factor a 3 first, then factor as usual.

$$\begin{aligned} &3x^2 - 24x + 45 \qquad \text{Factor the GCF} \\ &3(x^2 - 8x + 15) \end{aligned}$$

Next, we only concentrate on the expression in the parenthesis. What two numbers multiply to 15 that add up to -8?

p and q	Product	Sum
$3, 5$	15	8
$-3, -5$	15	-8

We can see from the table that -3 and -5 are the two numbers whose product is 15 and sum is -8. We use these two numbers to rewrite the expression in factored form:

$$3x^2 - 24x + 4$$

$$3(x-3)(x-5)$$

We can always verify the factored form by multiplying and obtaining the original expression.

Warning!

Students tend to forget to write the GCF in the final answer. Be sure to always include the GCF in the final factored form.

Also, to *factor completely*, it is required the GCF is factored out of the expression. If not, then the expression is not *factored completely*.

World Note

The first person to use letters for unknown values was *Francois Vieta* in 1591 in France. He used vowels to represent variables for solving, just as codes used letters to represent an unknown message.

7.2.3 Factoring Trinomials of the Form $x^2 + bx + c$ Homework

Factor completely.

1. $p^2 + 17p + 72$

2. $n^2 - 9n + 8$

3. $x^2 - 9x - 10$

4. $b^2 + 12b + 32$

5. $x^2 + 3x - 70$

6. $n^2 - 8n + 15$

7. $p^2 + 15p + 54$

8. $n^2 - 15n + 56$

9. $u^2 - 8uv + 15v^2$

10. $m^2 + 2mn - 8n^2$

11. $x^2 - 11xy + 18y^2$

12. $x^2 + xy - 12y^2$

13. $x^2 + 4xy - 12y^2$

14. $5a^2 + 60a + 100$

15. $6a^2 + 24a - 192$

16. $6x^2 + 18xy + 12y^2$

17. $6x^2 + 96xy + 378y^2$

18. $x^2 + x - 72$

19. $x^2 + x - 30$

20. $x^2 + 13x + 40$

21. $b^2 - 17b + 70$

22. $x^2 + 3x - 18$

23. $a^2 - 6a - 27$

24. $p^2 + 7p - 30$

25. $m^2 - 15mn + 50n^2$

26. $m^2 - 3mn - 40n^2$

27. $x^2 + 10xy + 16y^2$

28. $u^2 - 9uv + 14v^2$

29. $x^2 + 14xy + 45y^2$

30. $4x^2 + 52x + 168$

31. $5n^2 - 45n + 40$

32. $5v^2 + 20v - 25$

33. $5m^2 + 30mn - 90n^2$

34. $6m^2 - 36mn - 162n^2$

7.3 Factoring trinomials of the form $ax^2 + bx + c$

When factoring trinomials, we factored by grouping after we split the middle term. We continue to use this method for further factoring, like trinomials of the form $ax^2 + bx + c$, where a, b, and c are coefficients.

World Note

French philosopher *Rene Descartes* first used letters from the beginning of the alphabet to represent known values, a, b, c, and letters from the end to represent unknown values, x, y, z.

A simple example of a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, is $3x^2 + 11x + 6$. Recall, that this trinomial was a product of two binomials and we applied the FOIL method to obtain the result. Let's take a look at the binomial factors:

$$(3x+2)(x+3) = \underbrace{3x^2}_{\text{F}} + \underbrace{9x}_{\text{O}} + \underbrace{2x}_{\text{I}} + \underbrace{6}_{\text{L}} = 3x^2 + 11x + 6$$

In this section, we reverse the FOIL method and factor by grouping, or use the method of trial-and-error.

7.3.1 Factoring trinomials of the form $ax^2 + bx + c$ using grouping

Steps for factoring trinomials of the form $ax^2 + bx + c$

Step 1. Find two numbers, p and q, whose sum is b and product is $a \cdot c$.

Step 2. Rewrite the expression so that the middle term is split into two terms, p and q.

Step 3. Factor by grouping.

Step 4. Verify the factored form by finding the product.

Example 7.24

Factor: $3x^2 + 11x + 6$

Solution.

First we identify $a = 3$, $b = 11$ and $c = 6$. We ask ourselves,“ What two numbers multiply to $3 \cdot 6$ that add up to 11?”

Step 1. Find two numbers whose sum is 11 and product is 18:

p and q	Product	Sum
2,9	18	11
3,6	18	9
1,18	18	19

We can see from the table that 2 and 9 are the two numbers whose product is 18 and sum is 11. We use these two numbers in **Step 2.**

Step 2. Rewrite the expression so that the middle term is split into two terms, $2x$ and $9x$:

$$3x^2 + 11x + 6$$

$$3x^2 \underbrace{+2x + 9x}_{\text{sum is } 11x} + 6$$

Step 3. Factor by grouping.

$3x^2 + 2x + 9x + 6$	Group the first two terms and the last two terms
$(3x^2 + 2x) + (9x + 6)$	Factor x from the first group and 3 from the second group
$x(3x + 2) + 3(3x + 2)$	Factor the GCF $(3x + 2)$
$(3x + 2)(x + 3)$	Factored form

Step 4. Verify the factored form by finding the product:

$(3x + 2)(x + 3)$	FOIL
$3x^2 + 2x + 9x + 6$	Combine like terms
$3x^2 + 11x + 6$	✓Original expression

Thus, the factored form is $(3x + 2)(x + 3)$.

Example 7.25

Factor: $8x^2 - 2x - 15$

Solution.

First we identify $a = 8$, $b = -2$ and $c = -15$. We ask ourselves,“ What two numbers multiply to $8 \cdot -15$ that add up to -2?”

Step 1. Find two numbers whose sum is -2 and product is -120:

p and q	Product	Sum
$-8, 15$	-120	7
$8, -15$	-120	-7
$-10, 12$	-120	2
$10, -12$	-120	-2

We can see from the table that 10 and -12 are the two numbers whose product is -120 and sum is -2. We use these two numbers in **Step 2.**

Step 2. Rewrite the expression so that the middle term is split into two terms, $10x$ and $-12x$:

$$8x^2 - 2x - 15$$

$$8x^2 \underbrace{+10x - 12x}_{\text{sum is } -2x} - 15$$

Step 3. Factor by grouping.

$8x^2 + 10x - 12x - 15$	Group the first two terms and the last two terms
$(8x^2 + 10x) + (-12x - 15)$	Factor $2x$ from the first group and -3 from the second group
$2x(4x + 5) - 3(4x + 5)$	Factor the GCF $(4x + 5)$
$(4x + 5)(2x - 3)$	Factored form

Step 4. Verify the factored form by finding the product:

$$\begin{aligned} (4x+5)(2x-3) & \quad \text{FOIL} \\ 8x^2+10x-12x-15 & \quad \text{Combine like terms} \\ 8x^2-2x-15 & \quad \checkmark\text{Original expression} \end{aligned}$$

Thus, the factored form is $(4x+5)(2x-3)$.

Warning!

When factoring trinomials of the form ax^2+bx+c, there is *no shortcut* as we saw when we factored trinomials of the form x^2+bx+c. There is no way around this despite student efforts and desires for a shortcut. The closest we get to a shortcut is factoring by trial-and-error in which next we discuss.

Since $a \neq 1$, this makes factoring more involved and forces us to factor by grouping, sometimes called the *ac method*, or by trial-and-error.

7.3.2 Factoring trinomials of the form ax^2+bx+c using trial-and-error

Example 7.26

Factor: $10x^2-27x+5$

Solution.

Factoring by trial-and-error is just that- *trial and error*. Recall, FOIL: first, outer, inner, last. We know the first's product is $10x^2$ and the last's product is 5. Let's try different combinations and FOIL each combination to obtain the original expression. This sometimes works out faster than the *ac* method and sometimes not. Hence, trial and error.

binomials	FOIL	Yes or no?
$(5x+5)(2x+1)$	$10x^2+15x+5$	NO
$(2x+5)(5x+1)$	$10x^2+27x+5$	NO
$(2x+5)(5x-1)$	$10x^2+23x-5$	NO
$(2x-5)(5x-1)$	$10x^2-27x+5$	YES

Once we obtain the original expression we can stop taking combinations. We have found the factored form of the original expression:

$$(2x-5)(5x-1)$$

We can see that this method may not be the most time efficient unless, of course, we get lucky and obtain the the right combination quickly.

Helpful tips for factoring by trial-and-error

For factoring trinomials of the form $x^2 + bx + c$, here are some helpful tips when factoring by trial-and-error:

- Always look at the x^2 and c terms first and determine factors of these terms.
- According to the trinomial's signs of the terms, the binomial factors take on these signs:

trinomials	binomials	signs in binomial factors
$x^2 + mx + n$	$(x + p)(x + q)$	all positive
$x^2 - mx + n$	$(x - p)(x - q)$	all negative
$x^2 - mx - n$	$(x - p)(x + q)$	one positive, one negative
$x^2 + mx - n$	$(x - p)(x + q)$	one positive, one negative

Now, for the last two cases, the binomials with subtraction and addition is to be determined by the student when FOIL-ing each combination. Also note, m, n, p, q are all coefficients and numbers, respectively. In general, we can use these patterns of the signs in binomial factors for any trinomial form expression.

Example 7.27

Factor: $4x^2 - xy - 5y^2$

Solution.

We know the first's product is $4x^2$ and the last's product is $-5y^2$. Let's try different combinations and FOIL each combination to obtain the original expression. Since the signs of the last two terms are negative, this means we have case 3 from above and the binomial factors will have alternating signs.

binomials	FOIL	Yes or no?
$(2x + 5y)(2x - y)$	$4x^2 + 8xy - 5y^2$	NO
$(2x - 5y)(2x + y)$	$4x^2 - 8xy - 5y^2$	NO
$(4x + 5y)(x - y)$	$4x^2 + xy - 5y^2$	NO
$(4x - 5y)(x + y)$	$4x^2 - xy - 5y^2$	YES

Once we obtain the original expression we can stop taking combinations. We have found the factored form of the original expression:

$$(4x - 5y)(x + y)$$

7.3.3 Factoring trinomials of the form $ax^2 + bx + c$ with a greatest common factor

As always, when factoring, we will first look for a GCF, then factor as usual.

Example 7.28

Factor: $-18x^3 - 33x^2 + 30x$

Solution.

Notice all three terms have a common factor of $-3x$. We factor $-3x$ first, then factor as usual.

$$\begin{array}{ll} -18x^3 - 33x^2 + 30x & \text{Factor the GCF} \\ -3x(6x^2 + 11x - 10) & \end{array}$$

Next, we only concentrate on the expression in the parenthesis. Let's factor by trial-and-error. We know the first's product is $6x^2$ and the last's product is -10. Let's try different combinations and FOIL each combination to obtain the original expression. Since the signs of the last two terms are positive and negative, respectively, this means we have case 4 from above and the binomial factors will have alternating signs.

binomials	FOIL	Yes or no?
$(2x+5)(3x-2)$	$4x^2+11x-10$	NO
$(2x-5)(3x+2)$	$4x^2-11x-10$	YES

Once we obtain the original expression we can stop taking combinations. We have found the factored form of the original expression:

$$-3x(2x-5)(3x+2)$$

Recall, we include the GCF in the final answer or else the trinomial isn't *factored completely*. Note, if the ax^2 term is negative, then we always factor a negative because factoring becomes less challenging when the leading term is positive.

Note

In general, when given a trinomial where the leading coefficient is negative, like in Example 7.28, we treat the leading coefficient as a GCF and factor it out prior to factoring the trinomial, i.e., if $a<0$, then factor the negative out prior to factoring by grouping or trial-and-error.

Example 7.29

Factor: $3x^2+2x-7$

Solution.

Let's try to factor by the ac method. First we identify $a=3$, $b=2$ and $c=-7$. We ask ourselves," What two numbers multiply to $3\cdot-7$ that add up to 2?"

Step 1. Find two numbers whose sum is 2 and product is -21:

p and q	Product	Sum
$-3, 7$	-21	4
$3, -7$	-21	-4
$-1, 21$	-21	20
$1, -21$	-21	-20

We can see from the table that there aren't any factors of -21 whose sum is 2. We only obtain sums with 4 and 20's. In this case, we call this trinomial *prime*.

7.3.4 Factoring trinomials using substitution

Even though we are used to factoring expressions of the form ax^2+bx+c, we can apply these same factoring strategies to other trinomials of the same form where

$$a(\text{base})^2+b(\text{base})+c$$

Example 7.30

For example, factor $2(y+1)^2+3(y+1)-35$.

Solution.

We can see that the base $(y+1)$ is squared, then to the first power in the linear term. Let's allow $u = y+1$ and rewrite the expression in terms of u: $2\underbrace{(y+1)}_{u}^2 + 3\underbrace{(y+1)}_{u} - 35$

$$2u^2 + 3u - 35$$

Let's factor the expression in terms of u. Then **substitute** back $u = y+1$ in the end.

$2u^2 + 3u - 35$	Factor using trial-and-error
$(2u- \quad)(u+ \quad)$	Using the last case in trial-and-error, we know the signs will be $-, +$
$(2u-7)(u+5)$	Two factors whose product is -35 are $-7, 5$

Checking with FOIL:

$$2u^2 + 10u - 7u - 35$$

$$2u^2 + 3u - 35 \checkmark$$

Recall, $(2u-7)(u+5)$ is the factored form of the expression $2u^2 + 3u - 35$, not the original expression, $2(y+1)^2 + 3(y+1) - 35$. We used ***u*-substitution** to write the original expression in terms of u so we could easily factor. Lastly, we need to substitute back in $u = y+1$ into the factored expression $(2u-7)(u+5)$ and simplify:

$$\big(2(y+1) - 7\big)\big((y+1) + 5\big)$$

$$\big(2y + 2 - 7\big)\big(y + 1 + 5\big)$$

Simplifying each factor, we get the final factored expression in terms of the original variable y to be

$$(2y-5)(y+6)$$

Factoring expressions of the form $a(base)^2 + b(base) + c$

When given a polynomial expression of the form $a(\text{base})^2 + b(\text{base}) + c$, we can use the same factoring strategies to factor these expressions by using ***u*-substitution**, where

$$u = \text{ base}$$

☠ ***Warning!***

When using ***u*-substitution**, be sure to substitute back in the original base into the factored expression so that the final factored expression contains the original variable.

Example 7.31

Factor: $z^{2/3} + 2z^{1/3} - 80$

Solution.

We can see that the base is z. Furthermore, we can rewrite the first term as $\left(z^{1/3}\right)^2$, and the second term as $\left(z^{1/3}\right)^1 = z^{1/3}$. Let's allow $u = z^{1/3}$ and rewrite the expression in terms of u:

$$\underbrace{\left(z^{1/3}\right)^2}_{u} + 2\underbrace{z^{1/3}}_{u} - 80$$

$$u^2 + 2u - 80$$

Let's factor the expression in terms of u. Then **substitute** back $u = z^{1/3}$ in the end.

$u^2 + 2u - 80$	Factor using trial-and-error
$(u- \quad)(u+ \quad)$	Using the last case in trial-and-error, we know the signs will be $-, +$
$(u-8)(u+10)$	Two factors whose product is -80 are $-8, 10$

Checking with FOIL:

$$u^2 + 10u - 8u - 80$$

$$u^2 + 2u - 80 \checkmark$$

Recall, $(u-8)(u+10)$ is the factored form of the expression $u^2 + 2u - 80$, not the original expression, $z^{2/3} + 2z^{1/3} - 80$. Lastly, we need to substitute back in $u = z^{1/3}$ into the factored expression $(u-8)(u+10)$ and simplify:

$$\left(z^{1/3} - 8\right)\left(z^{1/3} + 10\right)$$

Simplifying each factor, we get the final factored expression in terms of the original variable z to be

$$\left(z^{1/3} - 8\right)\left(z^{1/3} + 10\right)$$

7.3.5 Factoring Trinomials of the Form $ax^2 + bx + c$ Homework

What are possible values of p given the trinomial $ax^2 + bx + c$?

1. $(2x+p)(x+3)$ given $2x^2+7x+3$

2. $(3x+p)(5x+2)$ given $15x^2-14x-8$

3. $(x+p)(4x-3)$ given $4x^2+x-3$

4. $(7x+p)(x-4)$ given $7x^2-30x+8$

Factor completely by grouping.

5. $7x^2-48x+36$

6. $7b^2+15b+2$

7. $5a^2-13a-28$

8. $2x^2-5x+2$

9. $2x^2+19x+35$

10. $2b^2-b-3$

11. $5k^2+13k+6$

12. $3x^2-17x+20$

13. $3x^2+17xy+10y^2$

14. $5x^2+28xy-49y^2$

15. $6x^2-39x-21$

16. $21k^2-87k-90$

17. $14x^2-60x+16$

18. $6x^2+29x+20$

19. $4k^2-17k+4$

20. $4x^2+9xy+2y^2$

Factor completely by trial-and-error.

21. $4m^2-9mn-9n^2$

22. $4x^2+13xy+3y^2$

23. $12x^2+62xy+70y^2$

24. $24x^2-52xy+8y^2$

25. $7n^2-44n+12$

26. $7v^2-24v-16$

27. $5n^2-4n-20$

28. $3r^2-4r-4$

29. $7x^2+29x-30$

30. $5k^2-26k+24$

31. $3r^2+16r+21$

32. $3u^2+13uv-10v^2$

33. $7x^2-2xy-5y^2$

34. $5u^2+31uv-28v^2$

35. $10a^2-54a-36$

36. $21n^2+45n-54$

Factor completely.

37. $4r^2+r-3$

38. $6p^2+11p-7$

39. $4r^2+3r-7$

40. $4m^2+6mn+6n^2$

41. $4x^2-6xy+30y^2$

42. $18u^2-3uv-36v^2$

43. $16x^2+60xy+36y^2$

44. $12x^2+50xy+28y^2$

Factor completely by using substitution.

45. $(x - 2y)^2 + 7(x - 2y) - 18$

46. $3(a - b)^2 - (a - b) - 44$

47. $(5a - 3b)^2 + 8(5a - 3b) + 16$

48. $(x - 4y)^2 - 10(x - 4y) + 25$

49. $(3r - 4)^2 - 4(3r - 4) - 12$

50. $(7m - 1)^2 + 12(7m - 1) - 45$

51. $w^{2/5} - 2w^{1/5} - 80$

52. $x - 5x^{1/2} - 50$

53. $x^{2/3} + 8x^{1/3} - 20$

54. $x^{6/7} - 10x^{3/7} + 21$

7.4 Special products

In the previous chapter, we recognized two special products: Difference of two squares and Perfect square trinomials. In this section, we discuss these special products to factor expressions.

Special products

- Difference of two squares: $a^2 - b^2 = (a+b)(a-b)$
- Difference of two fourth powers: $a^4 - b^4 = \left(a^2+b^2\right)(a+b)(a-b)$
- Perfect square trinomials: $a^2 - 2ab + b^2 = (a-b)^2$ or $a^2 + 2ab + b^2 = (a+b)^2$

7.4.1 Difference of two squares

Example 7.32

Factor completely: $x^2 - 16$

Solution.

Notice we have a difference of two squares. We can use the formula to factor:

$$\begin{array}{rl} x^2 - 16 & \text{Difference of two squares} \\ (x)^2 - (4)^2 & \text{Factor} \\ (x+4)(x-4) & \text{Factored form} \end{array}$$

As long as we determine a and b from the formula, we can easily write the expression in factored form. Let's continue with another example.

Example 7.33

Factor completely: $9a^2 - 25b^2$

Solution.

Notice we have a difference of two squares. We can use the formula to factor:

$$\begin{array}{rl} 9a^2 - 25b^2 & \text{Difference of two squares} \\ (3a)^2 - (5b)^2 & \text{Factor} \\ (3a+5b)(3a-5b) & \text{Factored form} \end{array}$$

☠ ***Warning!***

It is important to note that a *sum of squares*, e.g., $x^2 + y^2$, is not factorable. Hence, a sum of two squares is always prime unless there is a greatest common factor.

7.4.2 Difference of two fourth powers

The difference of two fourth powers is just a difference of two squares with the exception that there is an additional difference of two squares to be factored in order to factor completely.

Example 7.34

Factor completely: $a^4 - b^4$

Solution.

Notice we have a difference of two squares. We can use the formula to factor:

$$\begin{aligned} a^4 - b^4 &\quad \text{Difference of two squares} \\ \left(a^2\right)^2 - \left(b^2\right)^2 &\quad \text{Factor} \\ \left(a^2 + b^2\right)\left(a^2 - b^2\right) &\quad \text{Difference of two squares...again} \\ \left(a^2 + b^2\right)(a + b)(a - b) &\quad \text{Factored form} \end{aligned}$$

Example 7.35

Factor completely: $x^4 - 16$

Solution.

Notice we have a difference of two squares. We can use the formula to factor:

$$\begin{aligned} x^4 - 16 &\quad \text{Difference of two squares} \\ \left(x^2\right)^2 - \left(4\right)^2 &\quad \text{Factor} \\ \left(x^2 + 4\right)\left(x^2 - 4\right) &\quad \text{Difference of two squares...again} \\ \left(x^2 + 4\right)(x + 2)(x - 2) &\quad \text{Factored form} \end{aligned}$$

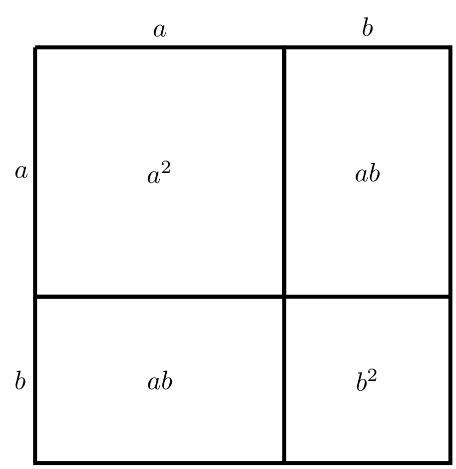
7.4.3 Perfect square trinomials

Students tend to wonder the reason for the word “square” in the formula when there are four sides to the geometric square shape. Well, it happens that these two are related. Let’s take a generic perfect square trinomial: $a^2 + 2ab + b^2$, and put this in a geometric representation:

Looking at the square to the right, the area of each square and rectangle are labeled within each figure. If we add the areas together, we would obtain

$$a^2 + 2ab + b^2$$

The reason we call this a perfect square trinomial is because the sum of all the areas is the area of the outer square.



Example 7.36

Factor completely: $x^2 - 6x + 9$

Solution.

Notice we have three terms. We could factor as usual, or recognize that this is a special product, a *perfect square trinomial*.

$$\begin{aligned} x^2 - 6x + 9 &\qquad \text{Perfect square trinomial} \\ (x)^2 - 2(x)(3) + (3)^2 &\qquad \text{Factor} \\ (x-3)^2 &\qquad \text{Factored form} \end{aligned}$$

Since the middle term of the trinomial was negative, then we have subtraction in the factored form.

Example 7.37

Factor completely: $4x^2 + 20xy + 25y^2$

Solution.

Notice we have three terms. We could factor as usual, or recognize that this is a special product, a *perfect square trinomial*.

$$\begin{aligned} 4x^2 + 20xy + 25y^2 &\qquad \text{Perfect square trinomial} \\ (2x)^2 + 2(2x)(5y) + (5y)^2 &\qquad \text{Factor} \\ (2x+5y)^2 &\qquad \text{Factored form} \end{aligned}$$

Since the middle term of the trinomial was positive, then we have addition in the factored form.

7.4.4 Factoring special products with a greatest common factor

Example 7.38

Factor completely: $72x^2 - 2$

Solution.

We have two terms and subtraction in between. Notice that this isn't a different of two squares, but that we do have a GCF. Let's factor the GCF and then see if we can use a special product formula:

$$\begin{aligned} 72x^2 - 2 &\qquad \text{Factor a GCF } 2 \\ 2(36x^2 - 1) &\qquad \text{Difference of two squares} \\ 2\big((6x)^2 - (1)^2\big) &\qquad \text{Factor} \\ 2(6x+1)(6x-1) &\qquad \text{Factored form} \end{aligned}$$

Note

In example 7.38, we couldn't factor using the difference of two squares right away. It wasn't until after we factored the GCF that we recognized the expression as a difference of two squares. Hence, we always look for a GCF to factor before applying any other factoring methods.

Example 7.39

Factor completely: $48x^2y - 24xy + 3y$

Solution.

Let's factor the GCF and then see if we can use a special product formula:

$$\begin{array}{rl} 48x^2y - 24xy + 3y & \text{Factor a GCF } 3y \\ 3y\big(16x^2 - 8x + 1\big) & \text{Perfect square trinomial} \\ 3y\big((4x)^2 - 2(4x)(1) + (1)^2\big) & \text{Factor} \\ 3y(4x-1)^2 & \text{Factored form} \end{array}$$

Since the middle term of the trinomial was negative, then we have subtraction in the factored form.

World Note

The first known record of work with polynomials comes from the Chinese around 200 BC. Problems would be written as "three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou. This would be the polynomial (trinomial) equation $3x + 2y + z = 29$.

7.4.5 A sum or difference of two cubes

Sum or difference of two cubes

There are special formulas for a sum or difference of two cubes.

- Difference of two cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- Sum of two cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

We can also use the acronym SOAP for the formulas for factoring a sum or difference of two cubes.

Same	binomial has the same sign as the expression
Opposite	middle term of the trinomial has the opposite sign than the expression
Always	
Positive	last term of the trinomial is always positive

SOAP is an easier way of remembering the signs in the formula because the formulas for the sum and difference of two cubes are the same except for the signs. Let's take a look:

$$a^3 \underbrace{-}_{\textit{sign}} b^3 = (a \underbrace{-}_{\textit{same}} b)(a^2 \underbrace{+}_{\textit{opposite}} ab \underbrace{+}_{\textit{positive}} b^2)$$

$$a^3 \underbrace{+}_{\textit{sign}} b^3 = (a \underbrace{+}_{\textit{same}} b)(a^2 \underbrace{-}_{\textit{opposite}} ab \underbrace{+}_{\textit{positive}} b^2)$$

Once we identify a and b, then we can just plug-n-chug into one of the formulas and use SOAP for the signs.

Example 7.40

Factor completely: $m^3 - 27$

Solution.

Notice we have a difference of two cubes. We can use the formula to factor:

$$\begin{array}{rl}
m^3 - 27 & \text{Difference of two cubes} \\
(m)^3 - (3)^3 & \text{Factor, where } a = m \text{ and } b = 3 \\
(m-3)\big((m)^2 + (3)(m) + (3)^2\big) & \text{Simplify} \\
(m-3)\big(m^2 + 3m + 9\big) & \text{Factored form}
\end{array}$$

Example 7.41

Factor completely: $125p^3 + 8r^3$

Solution.

Notice we have a sum of two cubes. We can use the formula to factor:

$$\begin{array}{rl}
125p^3 + 8r^3 & \text{Sum of two cubes} \\
(5p)^3 + (2r)^3 & \text{Factor, where } a = 5p \text{ and } b = 2r \\
(5p+2r)\big((5p)^2 - (5p)(2r) + (2r)^2\big) & \text{Simplify} \\
(5p+2r)\big(25p^2 - 10pr + 4r^2\big) & \text{Factored form}
\end{array}$$

Example 7.42

Factor completely: $128a^4b^2 + 54ab^5$

Solution.

Let's factor the GCF and then see if we can use a special product formula:

$$\begin{array}{rl}
128a^4b^2 + 54ab^5 & \text{Factor a GCF } 2ab^2 \\
2ab^2\big(64a^3 + 27b^3\big) & \text{Sum of two cubes} \\
2ab^2\big((4a)^3 + (3b)^3\big) & \text{Factor, where } a = 4a \text{ and } b = 3b \\
2ab^2(4a+3b)\big((4a)^2 - (4a)(3b) + (3b)^2\big) & \text{Simplify} \\
2ab^2(4a+3b)\big(16a^2 - 12ab + 9b^2\big) & \text{Factored form}
\end{array}$$

7.4.6 Special Products Homework

Factor completely by using the special product formulas.

1. $r^2 - 16$

2. $v^2 - 25$

3. $p^2 - 4$

4. $9k^2 - 4$

5. $3x^2 - 27$

6. $16x^2 - 36$

7. $18a^2 - 50b^2$

8. $a^2 - 2a + 1$

9. $x^2 + 6x + 9$

10. $x^2 - 6x + 9$

11. $25p^2 - 10p + 1$

12. $25a^2 + 30ab + 9b^2$

13. $4a^2 - 20ab + 25b^2$

14. $8x^2 - 24xy + 18y^2$

15. $8 - m^3$

16. $x^3 - 64$

17. $216 - u^3$

18. $125a^3 - 64$

19. $64x^3 + 27y^3$

20. $54x^3 + 250y^3$

21. $a^4 - 81$

22. $16 - z^4$

23. $x^4 - y^4$

24. $m^4 - 81b^4$

25. $x^2 - 9$

26. $x^2 - 1$

27. $4v^2 - 1$

28. $9a^2 - 1$

29. $5n^2 - 20$

30. $125x^2 + 45y^2$

31. $4m^2 + 64n^2$

32. $k^2 + 4k + 4$

33. $n^2 - 8n + 16$

34. $k^2 - 4k + 4$

35. $x^2 + 2x + 1$

36. $x^2 + 8xy + 16y^2$

37. $18m^2 - 24mn + 8n^2$

38. $20x^2 + 20xy + 5y^2$

39. $x^3 + 64$

40. $x^3 + 8$

41. $125x^3 - 216$

42. $64x^3 - 27$

43. $32m^3 - 108n^3$

44. $375m^3 + 648n^3$

45. $x^4 - 256$

46. $n^4 - 1$

47. $16a^4 - b^4$

48. $81c^4 - 16d^4$

7.5 Factoring, a general strategy

A general strategy to factoring

Step 1. Factor out the greatest common factor, if possible. This includes factoring a negative if the leading coefficient is negative.

Step 2. Determine the number of terms in the polynomial.

Step 3. **a)** Two Terms

- Difference of two squares: $a^2 - b^2 = (a+b)(a-b)$
- Difference of two cubes: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
- Sum of two cubes: $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
- Difference of two fourth powers: $a^4 - b^4 = \left(a^2+b^2\right)(a+b)(a-b)$

b) Three Terms

- Perfect square trinomial: $a^2+2ab+b^2 = (a+b)^2$ or $a^2-2ab+b^2=(a-b)^2$
- Old fashion way:
  - $x^2+(p+q)x+p\cdot q = (x+p)(x+q)$
  - $ax^2+bx+c \to$ Factor by grouping or by trial-and-error.

c) Four Terms

- Factor by grouping, rearranging terms, if needed.

Step 4. Check your work by FOIL or multiplying out the product of factors.

Example 7.43

Factor completely: $4x^2+56xy+196y^2$

Solution.

Let's first factor the GCF. Recall, there are three terms. So we can use either the perfect square trinomial formula or factor as usual by grouping or trial-and-error.

$$\begin{array}{rl} 4x^2+56xy+196y^2 & \text{Factor a GCF } 4 \\ 4\left(x^2+14xy+49y^2\right) & \text{Perfect square trinomial} \\ 4\left((x)^2+2(x)(7y)+(7y)^2\right) & \text{Factor, where } a=x \text{ and } b=7y \\ 4(x+7y)^2 & \text{Factored form} \end{array}$$

Example 7.44

Factor completely: $5x^2y+15xy-35x^2-105x$

Solution.

Let's first factor the GCF. Recall, there are four terms. So we can use factor by grouping.

$$\begin{array}{rl} 5x^2y+15xy-35x^2-105x & \text{Factor a GCF } 5x \\ 5x\left(xy+3y-7x-21\right) & \text{Factor by grouping} \\ 5x\left((xy+3y)+(-7x-21)\right) & \text{Factor the GCF from each group} \end{array}$$

$5x\big(y(x+3)-7(x+3)\big)$	Factor the GCF $(x+3)$
$5x(x+3)(y-7)$	Factored form

Example 7.45

Factor completely: $100x^2-400$

Solution.

Let's first factor the GCF. Recall, there is a difference of two terms. Since the variable x is squared, let's see if we can use the difference of two squares formula.

$100x^2-400$	Factor a GCF 100
$100(x^2-4)$	Difference of two squares
$100\big((x)^2-(2)^2\big)$	Factor
$100(x+2)(x-2)$	Factored form

Example 7.46

Factor: $108x^3y^2-39x^2y^2+3xy^2$

Solution.

Notice all three terms have a common factor of $3y^2$. We factor $3y^2$ first, then factor as usual or by using a special product.

$108x^3y^2-39x^2y^2+3xy^2$	Factor the GCF
$3xy^2\big(36x^2-13x+1\big)$	

Next, we only concentrate on the expression in the parenthesis. Let's factor by trial-and-error. We know the first's product is $36x^2$ and the last's product is 1. Since the signs of the last two terms are negative and positive, respectively, then the binomial factors will have negative signs.

binomials	FOIL	Yes or no?
$(6x-1)(6x-1)$	$36x^2-12x+1$	NO
$(18x-1)(2x-1)$	$36x^2-20x+1$	NO
$(9x-1)(4x-1)$	$36x^2-13x+1$	YES

We have found the factored form of the original expression:

$$3xy^2(9x-1)(4x-1)$$

World Note

Variables originated in ancient Greece where *Aristotle* would use a single capital letter to represent a number.

Example 7.47

Factor completely: $5+625y^3$

Solution.

Let's factor the GCF . Recall, there is a sum of two terms. Since the only formula with a sum of two terms is the sum of two cubes, then we most likely will be using this special product formula.

$5 + 625y^3$	Rewrite in standard form
$625y^3 + 5$	Factor a GCF 5
$5\big(125y^3 + 1\big)$	Sum of two cubes
$5\big((5y)^3 + (1)^3\big)$	Factor, where $a = 5y$ and $b = 1$
$5(5y + 1)\big((5y)^2 - (5y)(1) + (1)^2\big)$	Simplify
$5(5y + 1)\big(25y^2 - 5y + 1\big)$	Factored form

7.5.1 Factoring, A General Strategy Homework

Factor completely.

1. $24az - 18ah + 60yz - 45yh$

2. $5u^2 - 9uv + 4v^2$

3. $-2x^3 + 128y^3$

4. $5n^3 + 7n^2 - 6n$

5. $54u^3 - 16$

6. $n^2 - n$

7. $x^2 - 4xy + 3y^2$

8. $9x^2 - 25y^2$

9. $m^2 - 4n^2$

10. $36b^2c - 16xd - 24b^2d + 24xc$

11. $128 + 54x^3$

12. $2x^3 + 6x^2y - 20y^2x$

13. $n^3 + 7n^2 + 10n$

14. $27x^3 - 64$

15. $5x^2 + 2x$

16. $3k^3 - 27k^2 + 60k$

17. $mn - 12x + 3m - 4xn$

18. $16x^2 - 8xy + y^2$

19. $27m^2 - 48n^2$

20. $9x^3 + 21x^2y - 60y^2x$

21. $2m^2 + 6mn - 20n^2$

22. $2x^2 - 11x + 15$

23. $16x^2 + 48xy + 36y^2$

24. $20uv - 60u^3 - 5xv + 15xu^2$

25. $2x^3 + 5x^2y + 3y^2x$

26. $54 - 128x^3$

27. $5x^2 - 22x - 15$

28. $45u^2 - 150uv + 125v^2$

29. $x^3 - 27y^3$

30. $12ab - 18a + 6nb - 9n$

31. $3m^3 - 6m^2n - 24n^2m$

32. $64m^3 + 27n^3$

33. $3ac + 15ad^2 + x^2c + 5x^2d^2$

34. $64m^3 - n^3$

35. $16a^2 - 9b^2$

36. $2x^2 - 10x + 12$

37. $32x^2 - 18y^2$

38. $2k^2 + k - 10$

39. $v^2 + v$

40. $x^3 + 4x^2$

41. $9n^3 - 3n^2$

42. $2u^2v^2 - 11uv^3 + 15v^4$

7.6 Solve by factoring

When solving linear equations, such as $2x - 5 = 21$, we can solve by isolating the variable on one side and a number on the other side. However, in this chapter, we have an x^2 term, so if it looks different, *then it is different*. Hence, we need a new method for solving trinomial equations. One method is using the *zero product rule*. There are other methods for solving trinomial equations, but that is for a future chapter.

Definition

A ***polynomial equation*** is any equation that contains a polynomial expression. A ***trinomial equation*** is written in the form

$$ax^2 + bx + c = 0,$$

where a, b, c are coefficients, and $a \neq 0$.

7.6.1 Zero product rule

Zero product rule

$$a \cdot b = 0 \text{ implies } a = 0 \text{ or } b = 0, \text{ or both } a = b = 0$$

The zero product rule states that in order for a product to be zero, then one of its factors must be zero, or even both since $0 \cdot 0 = 0$. Let's extend this rule into solving for a trinomial equation.

Example 7.48

Solve for x: $(2x - 3)(5x + 1) = 0$

Solution.

Using the zero product rule, we know that in order for this product to be equal to zero, then at least one of the factors must be zero:

$$\begin{aligned} (2x-3)(5x+1) &= 0 && \text{Set each factor equal to zero} \\ 2x - 3 = 0 \quad \text{or} \quad 5x + 1 &= 0 && \text{Solve} \\ 2x = 3 \quad \text{or} \quad 5x &= -1 \\ x = \frac{3}{2} \quad \text{or} \quad x &= -\frac{1}{5} && \text{Solution} \end{aligned}$$

7.6.2 Solve by factoring

Steps for solving trinomial equations

Step 1. Write the given equation in the form $ax^2 + bx + c = 0$.

Step 2. Factor the left side of the equation into a product of factors.

Step 3. Use the zero product rule to set each factor equal to zero and then solve for the unknown.

Step 4. Verify the solution(s).

Example 7.49

Solve for x: $4x^2 + x - 3 = 0$

Solution.

Step 1. The equation is already given with zero on the right side.

$$4x^2 + x - 3 = 0$$

Step 2. Factor the left side of the equation into a product of factors:

$$\begin{aligned} 4x^2 + x - 3 &= 0 \\ 4x^2 - 3x + 4x - 3 &= 0 \\ x(4x-3) + 1(4x-3) &= 0 \\ (4x-3)(x+1) &= 0 \end{aligned}$$

Step 3. Use the zero product rule to set each factor equal to zero and then solve for the unknown:

$$\begin{aligned} (4x-3)(x+1) &= 0 && \text{Set each factor equal to zero} \\ 4x - 3 = 0 \quad \text{or} \quad x+1 &= 0 && \text{Solve} \\ 4x = 3 \quad \text{or} \quad x &= -1 && \\ x = \frac{3}{4} \quad \text{or} \quad x &= -1 && \text{Solution} \end{aligned}$$

Step 4. Verify the solution(s): $x = \frac{3}{4}$ and $x = -1$

$$\begin{aligned} 4\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right) - 3 &\stackrel{?}{=} 0 & \qquad 4(-1)^2 + (-1) - 3 &\stackrel{?}{=} 0 \\ 4 \cdot \frac{9}{16} + \frac{3}{4} - 3 &\stackrel{?}{=} 0 & 4(1) - 1 - 3 &\stackrel{?}{=} 0 \\ 0 &= 0 \quad \checkmark & 0 &= 0 \quad \checkmark \end{aligned}$$

Thus, the solutions are $x = \frac{3}{4}$ and $x = -1$.

7.6.3 Rewrite the equation with zero on one side

Example 7.50

Solve for x: $x^2 = 8x - 15$

Solution.

Step 1. Write the given equation in the form with zero on the right side:

$$\begin{aligned} x^2 &= 8x - 15 \\ x^2 - 8x + 15 &= 0 \end{aligned}$$

Step 2. Factor the left side of the equation into a product of factors:

$$\begin{aligned} x^2 - 8x + 15 &= 0 \\ (x-5)(x-3) &= 0 \end{aligned}$$

Step 3. Use the zero product rule to set each factor equal to zero and then solve for the unknown:

$$\begin{aligned} (x-5)(x-3) &= 0 && \text{Set each factor equal to zero} \\ x-5=0 \quad \text{or} \quad x-3 &= 0 && \text{Solve} \\ x=5 \quad \text{or} \quad x &= 3 && \text{Solution} \end{aligned}$$

Step 4. Verify the solution(s): $x = 5$ and $x = 3$

$$\begin{aligned} (5)^2 &\stackrel{?}{=} 8(5) - 15 & (3)^2 &\stackrel{?}{=} 8(3) - 15 \\ 25 &\stackrel{?}{=} 40 - 15 & 9 &\stackrel{?}{=} 24 - 15 \\ 25 &= 25 \quad \checkmark & 9 &= 9 \quad \checkmark \end{aligned}$$

Thus, the solutions are $x = 5$ and $x = 3$.

7.6.4 Simplify the equation

Sometimes the equation isn't so straightforward. We may have to do some preliminary work so that the equation takes the form of a trinomial equation and then we can use the zero product rule.

Example 7.51

Solve for x: $(x-7)(x+3) = -9$

Solution.

Step 1. Write the given equation in the form with zero on the right side. Notice, we will have to FOIL the left side first, then obtain zero on the right.

$$\begin{aligned} (x-7)(x+3) &= -9 \\ x^2 - 4x - 21 &= -9 \\ x^2 - 4x - 12 &= 0 \end{aligned}$$

Step 2. Factor the left side of the equation into a product of factors:

$$\begin{aligned} x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0 \end{aligned}$$

Step 3. Use the zero product rule to set each factor equal to zero and then solve for the unknown:

$$\begin{aligned} (x-6)(x+2) &= 0 && \text{Set each factor equal to zero} \\ x-6=0 \quad \text{or} \quad x+2 &= 0 && \text{Solve} \\ x=6 \quad \text{or} \quad x &= -2 && \text{Solution} \end{aligned}$$

Step 4. Verify the solution(s): $x = 6$ and $x = -2$

$$\begin{aligned} ((6)-7)((6)+3) &\stackrel{?}{=} -9 & ((-2)-7)((-2)+3) &\stackrel{?}{=} -9 \\ -9 &= -9 \quad \checkmark & -9 &= -9 \quad \checkmark \end{aligned}$$

Thus, the solutions are $x = 6$ and $x = -2$.

Example 7.52

Solve for x: $3x^2 + 4x - 5 = 7x^2 + 4x - 14$

Solution.

Step 1. Write the given equation in the form with zero on the right side. Notice, we will have to combine like terms to obtain zero on the right.

$$\begin{aligned} 3x^2+4x-5 &= 7x^2+4x-14 \\ -4x^2+9 &= 0 \\ (-1)\left(-4x^2+9\right) &= 0(-1) \\ 4x^2-9 &= 0 \end{aligned}$$

Step 2. Factor the left side of the equation into a product of factors:

$$\begin{aligned} 4x^2-9 &= 0 \\ (2x+3)(2x-3) &= 0 \end{aligned}$$

Step 3. Use the zero product rule to set each factor equal to zero and then solve for the unknown:

$$\begin{aligned} (2x+3)(2x-3) &= 0 && \text{Set each factor equal to zero} \\ 2x+3=0 \quad \text{or} \quad 2x-3 &= 0 && \text{Solve} \\ x=-\frac{3}{2} \quad \text{or} \quad x &= \frac{3}{2} && \text{Solution} \end{aligned}$$

Step 4. We leave verifying the solution(s): $x=-\dfrac{3}{2}$ and $x=\dfrac{3}{2}$, to the student.

Example 7.53

Solve for x: $4x^2=12x-9$

Solution.

Step 1. Write the given equation in the form with zero on the right side:

$$\begin{aligned} 4x^2 &= 12x-9 \\ 4x^2-12x+9 &= 0 \end{aligned}$$

Step 2. Factor the left side of the equation into a product of factors:

$$\begin{aligned} 4x^2-12x+9 &= 0 \\ (2x-3)^2 &= 0 \end{aligned}$$

Step 3. Use the zero product rule to set each factor equal to zero and then solve for the unknown:

$$\begin{aligned} (2x-3)^2 &= 0 && \text{Rewrite as two factors} \\ (2x-3)(2x-3) &= 0 && \text{Set each factor equal to zero} \\ 2x-3=0 \quad \text{or} \quad 2x-3 &= 0 && \text{Solve} \\ x=\frac{3}{2} \quad \text{or} \quad x &= \frac{3}{2} && \text{Solution} \end{aligned}$$

Notice we obtain the same solution for both factors. Even though we usually obtain two different solutions, in some cases, we obtain one solution. We call this solution *with multiplicity two.*

Step 4. We leave verifying the solution(s): $x = \frac{3}{2}$, to the student.

Thus, the solution is $x = \frac{3}{2}$ *with multiplicity two.*

Note

In solving trinomials of the form $ax^2 + bx + c = 0$, we should always obtain two solutions. There is one case in which we will obtain one solution *with multiplicity two.* This case is when the trinomial equation is a perfect square trinomial.

Example 7.54

Solve for x: $4x^2 = 8x$

Solution.

Step 1. Write the given equation in the form with zero on the right side.

$$\begin{aligned} 4x^2 &= 8x \\ 4x^2 - 8x &= 0 \end{aligned}$$

Step 2. Factor the left side of the equation into a product of factors. Notice here, we will only factor a GCF.

$$\begin{aligned} 4x^2 - 8x &= 0 \\ 4x(x-2) &= 0 \end{aligned}$$

Step 3. Use the zero product rule to set each factor equal to zero and then solve for the unknown:

$$\begin{aligned} 4x(x-2) &= 0 && \text{Set each factor equal to zero} \\ 4x = 0 \quad \text{or} \quad x - 2 &= 0 && \text{Solve} \\ x = 0 \quad \text{or} \quad x &= 2 && \text{Solution} \end{aligned}$$

Step 4. We leave verifying the solution(s): $x = 0$ and $x = 2$, to the student.

Example 7.55

Solve for x: $2x^3 - 14x^2 + 24x = 0$

Solution.

Step 1. We were given the equation in the form with zero on the right side:

$$2x^3 - 14x^2 + 24x = 0$$

Step 2. Factor the left side of the equation into a product of factors. Notice here, we will factor a

GCF in addition to factoring the trinomial.

$$\begin{aligned} 2x^3 - 14x^2 + 24x &= 0 \\ 2x\left(x^2 - 7x + 12\right) &= 0 \\ 2x(x-3)(x-4) &= 0 \end{aligned}$$

Step 3. Use the zero product rule to set each factor equal to zero and then solve for the unknown:

$$\begin{array}{rl} 2x(x-3)(x-4) = 0 & \text{Set each factor equal to zero} \\ 2x = 0 \quad \text{or} \quad x-3=0 \quad \text{or} \quad x-4=0 & \text{Solve} \\ x = 0 \quad \text{or} \quad x=3 \quad \text{or} \quad x=4 & \text{Solution} \end{array}$$

Step 4. We leave verifying the solution(s): $x = 0$, $x = 3$, and $x = 2$, to the student.

Notice, we obtained three solutions to the equation. Although we were given a trinomial, notice the degree of the trinomial was 3, i.e., when we factored, we obtained three factors. Hence, we will have three solutions. In general, the number of solutions will be at most the number of factors, e.g., we obtain two factors and one solution with multiplicity two.

World Note

While factoring works great to solve problems with an x^2 term, *Tartaglia*, in 16$^{\text{th}}$ century Italy, developed a method to solve problems with x^3. He kept his method a secret until another mathematician, *Cardan*, talked him out of his secret and published the results. To this day, the formula is known as Cardan's Formula.

7.6.5 Solve by Factoring Homework

Solve each equation by factoring.

1. $(k-7)(k+2)=0$

2. $(x-1)(x+4)=0$

3. $6x^2-150=0$

4. $2n^2+10n-28=0$

5. $7x^2+26x+15=0$

6. $5n^2-9n-2=0$

7. $x^2-4x-8=-8$

8. $x^2-5x-1=-5$

9. $49p^2+371p-163=5$

10. $7x^2+17x-20=-8$

11. $7r^2+84=-49r$

12. $x^2-6x=16$

13. $3v^2+7v=40$

14. $35x^2+120x=-45$

15. $4k^2+18k-23=6k-7$

16. $9x^2-46+7x=7x+8x^2+3$

17. $2m^2+19m+40=-2m$

18. $40p^2+183p-168=p+5p^2$

19. $(a+4)(a-3)=0$

20. $(2x+5)(x-7)=0$

21. $p^2+4p-32=0$

22. $m^2-m-30=0$

23. $40r^2-285r-280=0$

24. $2b^2-3b-2=0$

25. $v^2-8v-3=-3$

26. $a^2-6a+6=-2$

27. $7k^2+57k+13=5$

28. $4n^2-13n+8=5$

29. $7m^2-224=28m$

30. $7n^2-28n=0$

31. $6b^2=5+7b$

32. $9n^2+39n=-36$

33. $a^2+7a-9=-3+6a$

34. $x^2+10x+30=6$

35. $5n^2+41n+40=-2$

36. $24x^2+11x-80=3x$

37. $(x-3)^2-6(x-3)+8=0$

38. $4(2-5x)^2+8(2-5x)=-3$

7.7 Solve applications by factoring

There are two types of applications we will discuss in this section. There are many more types of problems we can model by a trinomial equation, but we will only discuss integer and rectangle problems for now. There's more to come in a later chapter.

7.7.1 Integer problems

Example 7.56

The product of two positive integer numbers is 48 and the sum of the same two numbers is 14. Find the numbers.

Solution.

First, we recall the method of substitution in system of equation in two variables. Recall, we solved for one variable in one equation, then substituted the expression into the second equation. We apply this method for integer problems. Let's set up the system. Let x and y be the two positive integers:

$$\begin{aligned} xy &= 48 \\ x + y &= 14 \end{aligned}$$

Taking the second equation and rewriting it as $y = 14 - x$, we substitute y into the first equation:

$$\begin{aligned} xy &= 48 \\ x(14 - x) &= 48 \end{aligned}$$

Now, we can solve.

$$\begin{aligned} x(14 - x) &= 48 && \text{Distribute} \\ 14x - x^2 &= 48 && \text{Rewrite with zero on the right side} \\ -x^2 + 14x - 48 &= 0 && \text{Multiply each term by } -1 \\ x^2 - 14x + 48 &= 0 && \text{Factor} \\ (x-6)(x-8) &= 0 && \text{Apply the zero product rule} \\ x - 6 = 0 \quad \text{or} \quad x - 8 &= 0 && \text{Solve} \\ x = 6 \quad \text{or} \quad x &= 8 && \text{Solution} \end{aligned}$$

Since both solutions are positive, then the numbers are 6 and 8.

World Note

William Horner, a British mathematician from the late 18^{th} century/early 19^{th} century, is credited with a method for solving simultaneous equations. However, Chinese mathematician *Chu Shih-chieh* in 1303 solved these equations with exponents as high as 14.

7.7.2 Rectangles using the area

Example 7.57

The length of a rectangle is 3 more inches than the width. If the area is 40 square inches, what are the dimensions?

Solution.

First, we need to recall the formula for the area of a rectangle:

$$A = \ell \cdot w$$

We use this formula to model a trinomial equation. We know that the length of the rectangle is 3 more inches than the width:

$$\ell = 3 + w$$

Next, we are given that the area of this rectangle is 40 square inches: $A = 40$. Let's model this information into a trinomial equation:

$$\begin{aligned} A &= \ell \cdot w && \text{Replace } \ell \text{ with } 3+w \text{ and } A=40 \\ 40 &= (3+w)\cdot w && \text{Distribute} \\ 40 &= 3w + w^2 && \text{Rewrite with zero on the right} \\ -w^2 - 3w + 40 &= 0 && \text{Multiply each term by } -1 \\ w^2 + 3w - 40 &= 0 \end{aligned}$$

Next, we solve the equation by factoring:

$$\begin{aligned} w^2 + 3w - 40 &= 0 && \text{Factor} \\ (w-5)(w+8) &= 0 && \text{Apply the zero product rule} \\ w-5=0 \quad \text{or} \quad w+8 &= 0 && \text{Solve} \\ w=5 \quad \text{or} \quad w &= -8 && \text{Solution} \end{aligned}$$

Since we have a rectangle and are finding the length and width of the rectangle, then we omit any negative solutions because length and width cannot be negative. Hence, we omit $w = -8$, and obtain a width of 5 inches and a length of 8 inches ($\ell = 3 + 5$).

7.7.3 Factoring applications with functions

Example 7.58

A rocket is launched at $t = 0$ seconds. Its height, in feet, above sea-level, as a function of time, t, is given by $h(t) = -16t^2 + 144t + 352$. When does the rocket hit the ground after it is launched?

Solution.

A rocket reaches the ground after it is launched when there is no distance between the rocket and the ground. Hence, the height between the rocket and ground is 0 feet. We need to find t

when $h(t) = 0$.

$$\begin{aligned} h(t) &= -16t^2 + 144t + 352 && \text{Replace } h(t) \text{ with zero} \\ 0 &= -16t^2 + 144t + 352 && \text{Factor the GCF } -16 \\ 0 &= -16\left(t^2 - 9t - 22\right) && \text{Divide each side by } -16 \\ 0 &= t^2 - 9t - 22 && \text{Factor} \\ 0 &= (t-11)(t+2) && \text{Apply the zero product rule} \\ t - 11 = 0 \quad &\text{or} \quad t + 2 = 0 && \text{Solve} \\ t = 11 \quad &\text{or} \quad t = -2 && \text{Solutions} \end{aligned}$$

With applications, we omit answers that are not reasonable and since we are trying to obtain the time it takes for the rocket to hit the ground, we should omit the solution $t = -2$. Thus, it will take 11 seconds for the rocket to hit the ground.

Example 7.59

The profit for a certain commodity, n, where n is in units, is given by the function

$$P(n) = -25n^2 + 400n + 1425$$

At the *break-even point*, the profit is zero, i.e., $P(n) = 0$. Find the number of units where the break-even point is located, i.e., find n when $P(n) = 0$.

Solution.

The *break-even point* is when the profit is zero, i.e., when $P(n) = 0$. We need to set $P(n) = 0$ and solve for n.

$$\begin{aligned} P(n) &= -25n^2 + 400n + 1425 && \text{Replace } P(n) \text{ with zero} \\ 0 &= -25n^2 + 400n + 1425 && \text{Factor the GCF } -25 \\ 0 &= -25\left(n^2 - 16n - 57\right) && \text{Divide each side by } -25 \\ 0 &= n^2 - 16n - 57 && \text{Factor} \\ 0 &= (n-19)(n+3) && \text{Apply the zero product rule} \\ n - 19 = 0 \quad &\text{or} \quad n + 3 = 0 && \text{Solve} \\ n = 19 \quad &\text{or} \quad n = -3 && \text{Solutions} \end{aligned}$$

With applications, we omit answers that are not reasonable and since we are trying to obtain the number of units where the break-even point is located, then we should omit the solution $n = -3$. Thus, the break-even point is located after 19 units are sold and produced.

7.7.4 Solve Applications by Factoring Homework

1. The product of two positive integer numbers is 104 and the sum of the same two numbers is 21. Find the numbers.

2. The product of two positive integer numbers is 50 and the sum of the same two numbers is 15. Find the numbers.

3. The product of two positive integer numbers is 48 and the sum of the same two numbers is 14. Find the numbers.

4. The product of two positive integer numbers is 30 and the sum of the same two numbers is 11. Find the numbers.

5. The product of two positive integer numbers is 44 and the sum of the same two numbers is 15. Find the numbers.

6. The product of two positive integer numbers is 18 and the sum of the same two numbers is 11. Find the numbers.

7. The product of two positive integer numbers is 28 and the sum of the same two numbers is 11. Find the numbers.

8. The product of two positive integer numbers is 18 and the sum of the same two numbers is 9. Find the numbers.

9. The product of two positive integer numbers is 54 and the sum of the same two numbers is 15. Find the numbers.

10. The length of a rectangle is 6 inches longer than it is wide. If the area is 160 square inches, what are the dimensions of the rectangle?

11. The length of a rectangle is 4 inches longer than it is wide. If the area is 117 square inches, what are the dimensions of the rectangle?

12. The length of a rectangle is 6 inches longer than it is wide. If the area is 91 square inches, what are the dimensions of the rectangle?

13. The length of a rectangle is 4 inches longer than it is wide. If the area is 45 square inches, what are the dimensions of the rectangle?

14. The length of a rectangle is 7 inches longer than it is wide. If the area is 98 square inches, what are the dimensions of the rectangle?

15. The length of a rectangle is 6 inches longer than it is wide. If the area is 40 square inches, what are the dimensions of the rectangle?

16. The length of a rectangle is 7 inches longer than it is wide. If the area is 144 square inches, what are the dimensions of the rectangle?

17. The length of a rectangle is 4 inches longer than it is wide. If the area is 96 square inches, what are the dimensions of the rectangle?

18. The length of a rectangle is 7 inches longer than it is wide. If the area is 60 square inches, what are the dimensions of the rectangle?

19. The length of a rectangle is 3 inches longer than it is wide. If the area is 88 square inches, what are the dimensions of the rectangle?

20. A rocket is launched at $t = 0$ seconds. Its height, in feet, above sea-level, as a function of time, t, is given by

$$h(t) = -16t^2 + 64t + 192$$

When does the rocket hit the ground after it is launched?

21. A rocket is launched at $t = 0$ seconds. Its height, in feet, above sea-level, as a function of time, t, is given by

$$h(t) = -16t^2 + 48t + 448$$

When does the rocket hit the ground after it is launched?

22. The profit for a certain commodity, n, where n is in units, is given by the function

$$P(n) = -25n^2 + 325n + 750$$

At the break-even point, the profit is zero, i.e., $P(n) = 0$. Find the number of units where the break-even point is located, i.e., find n when $P(n) = 0$.

23. The profit for a certain commodity, n, where n is in units, is given by the function

$$P(n) = -25n^2 + 325n + 1200$$

At the break-even point, the profit is zero, i.e., $P(n) = 0$. Find the number of units where the break-even point is located, i.e., find n when $P(n) = 0$.

7.8 Factoring Expressions & Solving by Factoring: Answers to the Homework Exercises

GCF and Grouping

1. $9 + 8b^2$
3. $7(8 - 5p)$
5. $-3a^2b(1 - 2ab)$
7. $10(2x^4 - 3x + 3)$
9. $5(6b^9 + ab - 3a^2)$
11. $5x^3y^2z(4x^5z + 3x^2 + 7y)$
13. $5q(6pr - p + 1)$
15. $-10x^{11}(4 + 2x - 5x^2 + 5x^3)$
17. $x - 5$
19. $10(5x - 8y)$
21. $4x^3(2y^2 + 1)$
23. $3(7p^6 + 10p^2 + 9)$
25. $3y^2(9y^5 + 4x + 3)$
27. $3(p + 4q - 5q^2r^2)$
29. $7b(4 + 2b + 5b^2 + b^4)$
31. $-4x^2(6x^4 + x^2 - 3x - 1)$
33. $(8r^2 - 5)(5r - 1)$
35. $(3b^2 - 7)(5b + 7)$
37. $(7x^2 - 4)(5x - 4)$
39. $(8x + 3)(4y + 5x)$
41. $(2x + 7y^2)(y - 4x)$
43. $(4u + 3)(8v - 5)$
45. $(3u - 7)(v - 2u)$
47. $(5x^2 - 8)(7x - 2)$
49. $(6x^2 + 5)(x - 8)$
51. $(7n^2 - 5)(n + 3)$
53. $(3a + b^2)(5b - 2)$
55. $(m - 5)(5n + 2)$
57. $2(u + 3)(2v + 7u)$
59. $(7a - 2)(8b - 7)$

Factoring Trinomials of the Form $x^2 + bx + c$

1. $(p + 9)(p + 8)$
3. $(x + 1)(x - 10)$
5. $(x - 7)(x + 10)$
7. $(p + 6)(p + 9)$
9. $(u - 5v)(u - 3v)$
11. $(x - 9y)(x - 2y)$
13. $(x + 6y)(x - 2y)$
15. $6(a - 4)(a + 8)$
17. $6(x + 9y)(x + 7y)$
19. $(x - 5)(x + 6)$
21. $(b - 10)(b - 7)$
23. $(a + 3)(a - 9)$
25. $(m - 5n)(m - 10n)$
27. $(x + 8y)(x + 2y)$
29. $(x + 5y)(x + 9y)$
31. $5(n - 8)(n - 1)$
33. $5(m^2 + 6mn - 18n^2)$

Factoring Trinomials of the Form $ax^2 + bx + c$

1. $p = 1$
3. $p = 1$
5. $(7x - 6)(x - 6)$
7. $(5a + 7)(a - 4)$
9. $(2x + 5)(x + 7)$
11. $(5k + 3)(k + 2)$
13. $(3x + 2y)(x + 5y)$
15. $3(2x + 1)(x - 7)$
17. $2(7x - 2)(x - 4)$
19. $(k - 4)(4k - 1)$
21. $(m - 3n)(4m + 3n)$
23. $2(2x + 7y)(3x + 5y)$
25. $(7n - 2)(n - 6)$
27. Prime, not factorable
29. $(7x - 6)(x + 5)$
31. $(3r + 7)(r + 3)$
33. $(7x + 5y)(x - y)$
35. $2(5a + 3)(a - 6)$
37. $(r + 1)(4r - 3)$
39. $(r - 1)(4r + 7)$
41. $2(2x^2 - 3xy + 15y^2)$
43. $4(x + 3y)(4x + 3y)$
45. $(x - 2y + 9)(x - 2y - 2)$
47. $(5a - 3b + 4)(5a - 3b + 4)$
49. $\left(w^{1/5} + 8\right)\left(w^{1/5} - 10\right)$
51. $\left(x^{1/3} + 10\right)\left(x^{1/3} - 2\right)$

Special Products

1. $(r+4)(r-4)$
3. $(p+2)(p-2)$
5. $3(x+3)(x-3)$
7. $2(3a+5b)(3a-5b)$
9. $(x+3)^2$
11. $(5p-1)^2$
13. $(2a-5b)^2$
15. $(2-m)(4+2m+m^2)$
17. $(6-u)(36+6u+u^2)$
19. $(4x+3y)(16x^2-12xy+9y^2)$
21. $(a^2+9)(a+3)(a-3)$
23. $(x^2+y^2)(x+y)(x-y)$
25. $(x+3)(x-3)$
27. $(2v+1)(2v-1)$
29. $5(n+2)(n-2)$
31. $4(m^2+16n^2)$
33. $(n-4)^2$
35. $(x+1)^2$
37. $2(3m-2n)^2$
39. $(x+4)(x^2-4x+16)$
41. $(5x-6)(25x^2+30x+36)$
43. $4(2m-3n)(4m^2+6mn+9n^2)$
45. $(x^2+16)(x+4)(x-4)$
47. $(4a^2+b^2)(2a+b)(2a-b)$

Factoring, A General Strategy

1. $3(2a+5y)(4z-3h)$
3. $-2(x-4y)(x^2+4xy+16y^2)$
5. $2(3u-2)(9u^2+6u+4)$
7. $(x-3y)(x-y)$
9. $(m+2n)(m-2n)$
11. $2(4+3x)(16-12x+9x^2)$
13. $n(n+2)(n+5)$
15. $x(5x+2)$
17. $(m-4x)(n+3)$
19. $3(3m+4n)(3m-4n)$
21. $2(m-2n)(m+5n)$
23. $4(2x+3y)^2$
25. $x(2x+3y)(x+y)$
27. $(5x+3)(x-5)$
29. $(x-3y)(x^2+3xy+9y^2)$
31. $3m(m+2n)(m-4n)$
33. $(3a+x^2)(c+5d^2)$
35. $(4a+3b)(4a-3b)$
37. $2(4x+3y)(4x-3y)$
39. $v(v+1)$
41. $3n^2(3n-1)$

Solve by Factoring

1. $7, -2$
3. $-5, 5$
5. $-\frac{5}{7}, -3$
7. $4, 0$
9. $\frac{3}{7}, -8$
11. $-4, -3$
13. $\frac{8}{3}, -5$
15. $-4, 1$
17. $-\frac{5}{2}, -8$
19. $-4, 3$
21. $4, -8$
23. $-\frac{7}{8}, 8$
25. $8, 0$
27. $-\frac{1}{7}, -8$
29. $8, -4$
31. $-\frac{1}{2}, \frac{5}{3}$
33. $2, -3$
35. $-\frac{6}{5}, -7$
37. $5, 7$

Solve Applications by Factoring

1. $8, 13$
3. $6, 8$
5. $4, 11$
7. $4, 7$
9. $6, 9$
11. 13×9 inches
13. 9×5 inches
15. 10×4 inches
17. 12×8 inches
19. 11×8 inches
21. 7 seconds
23. 16 units

Chapter 8

Rational Expressions

Chapter Objectives

By the end of this chapter, the student should be able to

- Evaluate rational expressions
- Obtain the excluded values of the expression
- Reduce rational expressions
- Multiply and divide rational expressions with and without factoring
- Find least common denominators
- Add and subtract rational expressions with and without common denominators
- Simplify compound rational expressions

Now that we discussed factoring to great lengths, we can introduce rational expressions where fractions contain polynomials. This chapter prepares us for future chapters when we solve rational equations and functions. Similar to fractions in arithmetic, we are concerned when the denominator is zero. Hence, we start this chapter evaluating rational expressions and determining the location of rational expressions' excluded values.

8.1 Reduce rational expressions

Definition

A ***rational expression*** is a ratio of two polynomials, i.e., a fraction where the numerator and denominator are polynomials.

8.1.1 Evaluate rational expressions

Example 8.1

Evaluate $\dfrac{x^2-4}{x^2+6x+8}$ when $x=-6$.

Solution.

$$\frac{x^2-4}{x^2+6x+8} \qquad \text{Plug-n-chug } x=-6$$

$$\frac{(-6)^2-4}{(-6)^2+6(-6)+8} \qquad \text{Simplify each numerator and denominator}$$

$$\frac{36-4}{36-36+8} \qquad \text{Simplify}$$

$$\frac{32}{8} \qquad \text{Reduce}$$

$$4 \qquad \text{Evaluated value}$$

8.1.2 Find excluded values of rational expressions

Rational expressions are special types of fractions, but still hold the same arithmetic properties. One property of fractions we recall is that the fraction is undefined when the denominator is zero.

Note

A rational expression is undefined where the denominator is zero.

Determine the excluded value(s) of a rational expression

Step 1. Set the denominator of the rational expression equal to zero.

Step 2. Solve the equation for the given variable.

Step 3. The values found in the previous step are the values excluded from the expression.

Example 8.2

Find the excluded value(s) of the expression: $\dfrac{-3z}{z+5}$

Solution.

Step 1. Set the denominator of the rational expression equal to zero:

$$z+5=0$$

Step 2. Solve the equation for z:

$$\begin{aligned} z+5&=0\\ z&=-5 \end{aligned}$$

Step 3. The values found in the previous step are the values excluded from the expression. Hence, the excluded value is $z=-5$.

Example 8.3

Find the excluded value(s) of the expression: $\dfrac{x^2-1}{3x^2+5x}$

Solution.

Step 1. Set the denominator of the rational expression equal to zero:

$$3x^2 + 5x = 0$$

Step 2. Solve the equation for x:

$$\begin{aligned} 3x^2 + 5x &= 0 \\ x(3x+5) &= 0 \\ x = 0 \quad &\text{or} \quad 3x+5=0 \\ x = 0 \quad &\text{or} \quad 3x = -5 \\ x = 0 \quad &\text{or} \quad x = -\frac{5}{3} \end{aligned}$$

Step 3. The values found in the previous step are the values excluded from the expression. Hence, the excluded values are $x = 0$ and $x = -5$.

Recall, the excluded values are values in which make the expression undefined. Hence, when evaluating rational expressions, we can evaluate the expressions for any values *except the excluded values*.

World Note

The number zero was not widely accepted in mathematical thought around the world for many years. It was the Mayans of Central America who first used zero to aid in the use of their base-20 system as a place holder.

8.1.3 Reduce rational expressions with monomials

Rational expressions are reduced, just as in arithmetic, even without knowing the value of the variable. When we reduce, we divide out common factors as we discussed with polynomial division with monomials. Now, we use factoring techniques and exponent properties to reduce rational expressions.

Reducing rational expressions

If P, Q, K are non-zero polynomials and $\dfrac{PK}{QK}$ is a rational expression, then

$$\frac{P \cdot \cancel{K}}{Q \cdot \cancel{K}} = \frac{P}{Q}$$

We call a rational expression ***irreducible*** if there are no more common factors among the numerator and denominator.

Example 8.4

Simplify: $\dfrac{15x^4y^2}{25x^2y^6}$

Solution.

Since the denominator is a monomial, then we reduce as usual and apply exponent rules:

$$\frac{15x^4y^2}{25x^2y^6} \qquad \text{Reduce by applying exponent rules}$$

$$\frac{3x^2}{5y^4} \qquad \text{Reduced expression}$$

8.1.4 Reduce rational expressions with polynomials

However, if there is a sum or difference in either the numerator or denominator, we first factor the numerator and denominator to obtain a product of factors, then reduce.

Example 8.5

Simplify: $\dfrac{28}{8x^2-16}$

Solution.

Since we have a difference in the denominator, we factor the denominator and then reduce.

$$\frac{28}{8x^2-16} \qquad \text{Factor a GCF 8 from the denominator}$$

$$\frac{\cancel{4}\cdot 7}{2\cdot\cancel{4}(x^2-2)} \qquad \text{Reduce by a factor of 4}$$

$$\frac{7}{2(x^2-2)} \qquad \text{Reduced expression}$$

Example 8.6

Simplify: $\dfrac{9x-3}{18x-6}$

Solution.

Since we have a difference in the denominator and numerator, we factor the denominator and numerator, and then reduce.

$$\frac{9x-3}{18x-6} \qquad \text{Factor the GCF from numerator and denominator}$$

$$\frac{3(3x-1)}{6(3x-1)} \qquad \text{Reduce by a factor of } 3(3x-1)$$

$$\frac{\cancel{3}\cancel{(3x-1)}}{2\cdot\cancel{3}\cdot\cancel{(3x-1)}} \qquad \text{Rewrite the expression}$$

$$\frac{1}{2} \qquad \text{Reduced expression}$$

Example 8.7

Simplify: $\dfrac{x^2-25}{x^2+8x+15}$

Solution.

Since we have a sum and difference of terms in the denominator and numerator, we factor the denominator and numerator, and then reduce.

$$\frac{x^2-25}{x^2+8x+15} \qquad \text{Factor using factoring tecniques}$$

$$\frac{(x+5)(x-5)}{(x+3)(x+5)} \qquad \text{Reduce by a factor of } (x+5)$$

$$\frac{\cancel{(x+5)}(x-5)}{(x+3)\cancel{(x+5)}} \qquad \text{Rewrite the expression}$$

$$\frac{x-5}{x+3} \qquad \text{Reduced expression}$$

☠ *Warning!*

We cannot reduce terms, *only factors*. This means we cannot reduce anything with a $+$ or $-$ between the parts. In example 8.7, we obtained the reduced expression $\frac{x-5}{x+3}$. Note, we are not allowed to divide out the x's because they are terms (separated by $+$ or $-$) not factors (separated by multiplication).

8.1.5 Reduce Rational Expressions Homework

Evaluate the expression for the given value.

1. $\dfrac{4v+2}{6}$ when $v=4$

2. $\dfrac{x-3}{x^2-4x+3}$ when $x=-4$

3. $\dfrac{b+2}{b^2+4b+4}$ when $b=0$

4. $\dfrac{b-3}{3b-9}$ when $b=-2$

5. $\dfrac{a+2}{a^2+3a+2}$ when $a=-1$

6. $\dfrac{n^2-n-6}{n-3}$ when $n=4$

Find the excluded value(s).

7. $\dfrac{3k^2+30k}{k+10}$

8. $\dfrac{15n^2}{10n+25}$

9. $\dfrac{10m^2+8m}{10m}$

10. $\dfrac{r^2+3r+2}{5r+10}$

11. $\dfrac{b^2+12b+32}{b^2+4b-32}$

12. $\dfrac{27p}{18p^2-36p}$

13. $\dfrac{x+10}{8x^2+80x}$

14. $\dfrac{10x+16}{6x+20}$

15. $\dfrac{6n^2-21n}{6n^2+3n}$

Simplify each expression.

16. $\dfrac{21x^2}{18x}$

17. $\dfrac{24a}{40a^2}$

18. $\dfrac{32x^3}{8x^4}$

19. $\dfrac{18m-24}{60}$

20. $\dfrac{20}{4p+2}$

21. $\dfrac{x+1}{x^2+8x+7}$

22. $\dfrac{32x^2}{28x^2+28x}$

23. $\dfrac{n^2+4n-12}{n^2-7n+10}$

24. $\dfrac{9v+54}{v^2-4v-60}$

25. $\dfrac{12x^2-42x}{30x^2-42x}$

26. $\dfrac{6a-10}{10a+4}$

27. $\dfrac{2n^2+19n-10}{9n+90}$

28. $\dfrac{21k}{24k^2}$

29. $\dfrac{90x^2}{20x}$

30. $\dfrac{10}{81n^3+36n^2}$

31. $\dfrac{n-9}{9n-81}$

32. $\dfrac{28m+12}{36}$

33. $\dfrac{49r+56}{56r}$

34. $\dfrac{b^2+14b+48}{b^2+15b+56}$

35. $\dfrac{30x-90}{50x+40}$

36. $\dfrac{k^2-12k+32}{k^2-64}$

37. $\dfrac{9p+18}{p^2+4p+4}$

38. $\dfrac{3x^2-29x+40}{5x^2-30x-80}$

39. $\dfrac{8m+16}{20m-12}$

40. $\dfrac{2x^2-10x+8}{3x^2-7x+4}$

41. $\dfrac{7n^2-32n+16}{4n-16}$

42. $\dfrac{n^2-2n+1}{6n+6}$

43. $\dfrac{7a^2-26a-45}{6a^2-34a+20}$

44. $\dfrac{56x-48}{24x^2+56x+32}$

45. $\dfrac{50b-80}{50b+20}$

46. $\dfrac{35v+35}{21v+7}$

47. $\dfrac{56x-48}{24x^2+56x+32}$

48. $\dfrac{4k^3-2k^2-2k}{9k^3-18k^2+9k}$

8.2 Multiply and divide rational expressions

We use the same method for multiplying and dividing fractions to multiply and divide rational expressions.

8.2.1 Multiply and divide rational expressions with monomials

Recall. When we multiply two fractions, we divide out the common factors, e.g.,

$$\frac{10}{9}\cdot\frac{21}{25}=\frac{\cancel{5}\cdot 2}{\cancel{3}\cdot 3}\cdot\frac{7\cdot\cancel{3}}{\cancel{5}\cdot 5}=\frac{14}{15}$$

We multiply rational expressions using the same method.

Example 8.8

Multiply: $\dfrac{25x^2}{9y^8}\cdot\dfrac{24y^4}{55x^7}$

Solution.

Since this is a product of a quotient of monomials, we reduce out common factors and use the rules of exponents.

$$\begin{array}{rl}
\dfrac{25x^2}{9y^8}\cdot\dfrac{24y^4}{55x^7} & \text{Multiply across numerators and denominators}\\[2ex]
\dfrac{25x^2\cdot 24y^4}{9y^8\cdot 55x^7} & \text{Rewrite grouping like-factors}\\[2ex]
\dfrac{25\cdot 24\cdot x^2\cdot y^4}{9\cdot 55\cdot x^7\cdot y^8} & \text{Reduce out common factors}\\[2ex]
\dfrac{5\cdot 8}{11\cdot 3\cdot x^5\cdot y^4} & \text{Multiply}\\[2ex]
\dfrac{40}{33x^5y^4} & \text{Product}
\end{array}$$

Recall. When we divide two fractions, we change the operation to multiplication and form the reciprocal of the second fraction. Then we multiply the fractions as we did before. E.g.,

$$\frac{7}{5}\div\frac{14}{15}=\frac{7}{5}\cdot\frac{15}{14}=\frac{\cancel{7}}{\cancel{5}}\cdot\frac{3\cdot\cancel{5}}{\cancel{7}\cdot 2}=\frac{3}{2}$$

We divide rational expressions using the same method.

Example 8.9

Divide: $\dfrac{a^4b^2}{a}\div\dfrac{b^4}{4}$

Solution.

Since this is a quotient of a quotient of monomials, we form the reciprocal of the second fraction and change the division to multiplication, reduce out common factors, and use the rules of exponents.

$$\frac{a^4b^2}{a} \div \frac{b^4}{4} \qquad \text{Rewrite the second fraction as its reciprocal}$$

$$\frac{a^4b^2}{a} \cdot \frac{4}{b^4} \qquad \text{Multiply across numerators and denominators}$$

$$\frac{4a^4b^2}{ab^4} \qquad \text{Reduce out common factors}$$

$$\frac{4a^3}{b^2} \qquad \text{Quotient}$$

8.2.2 Multiply and divide rational expressions with polynomials

When multiplying or dividing polynomials in rational expressions, we first factor using factoring techniques, then reduce out the common factors.

> **Warning!**
>
> We are not allowed to reduce terms, *only factors*.

Example 8.10

Multiply: $\dfrac{x^2-9}{x^2+x-20} \cdot \dfrac{x^2-8x+16}{3x+9}$

Solution.

Since we have polynomials in the numerators and denominators, we first factor, then reduce.

$$\frac{x^2-9}{x^2+x-20} \cdot \frac{x^2-8x+16}{3x+9} \qquad \text{Factor each numerator and denominator}$$

$$\frac{(x+3)(x-3)}{(x-4)(x+5)} \cdot \frac{(x-4)(x-4)}{3(x+3)} \qquad \text{Reduce out common factors}$$

$$\frac{\cancel{(x+3)}(x-3)}{\cancel{(x-4)}(x+5)} \cdot \frac{(x-4)\cancel{(x-4)}}{3\cancel{(x+3)}} \qquad \text{Rewrite}$$

$$\frac{(x-3)}{(x+5)} \cdot \frac{(x-4)}{3} \qquad \text{Multiply}$$

$$\frac{(x-3)(x-4)}{3(x+5)} \qquad \text{Product}$$

We can leave the product in factored form. There's no reason to multiply out the final answer unless an instructor requests the product that way.

Example 8.11

Divide: $\dfrac{x^2-x-12}{x^2-2x-8} \div \dfrac{5x^2+15x}{x^2+x-2}$

Solution.

Since we have division with polynomials in the numerators and denominators, we form the reciprocal of the second fraction and change the division to multiplication, factor, then reduce out common factors.

$$\frac{x^2-x-12}{x^2-2x-8} \div \frac{5x^2+15x}{x^2+x-2} \quad \text{Rewrite the second fraction as its reciprocal}$$

$$\frac{x^2-x-12}{x^2-2x-8} \cdot \frac{x^2+x-2}{5x^2+15x} \quad \text{Factor each numerator and denominator}$$

$$\frac{(x-4)(x+3)}{(x+2)(x-4)} \cdot \frac{(x+2)(x-1)}{5x(x+3)} \quad \text{Reduce out common factors}$$

$$\frac{\cancel{(x-4)}\cancel{(x+3)}}{\cancel{(x+2)}\cancel{(x-4)}} \cdot \frac{\cancel{(x+2)}(x-1)}{5x\cancel{(x+3)}} \quad \text{Rewrite}$$

$$\frac{1}{1} \cdot \frac{x-1}{5x} \quad \text{Multiply}$$

$$\frac{(x-1)}{5x} \quad \text{Quotient}$$

8.2.3 Multiply and divide rational expressions in general

We can combine multiplying and dividing rational expressions in one expression, but, remember, we form the reciprocal of the fraction that directly proceeds the division sign and then change the division to multiplication. Lastly, we can reduce the common factors.

> ☠ ***Warning!***
>
> We are not allowed to reduce terms, *only factors*.

Example 8.12

Simplify: $\dfrac{a^2+7a+10}{a^2+6a+5} \cdot \dfrac{a+1}{a^2+4a+4} \div \dfrac{a-1}{a+2}$

Solution.

$$\frac{a^2+7a+10}{a^2+6a+5} \cdot \frac{a+1}{a^2+4a+4} \div \frac{a-1}{a+2} \quad \text{Form the reciprocal of the last fraction}$$

$$\frac{a^2+7a+10}{a^2+6a+5} \cdot \frac{a+1}{a^2+4a+4} \cdot \frac{a+2}{a-1} \quad \text{Factor each numerator and denominator}$$

$$\frac{(a+5)(a+2)}{(a+5)(a+1)} \cdot \frac{(a+1)}{(a+2)(a+2)} \cdot \frac{(a+2)}{(a-1)} \quad \text{Reduce out common factors}$$

$$\frac{\cancel{(a+5)}\cancel{(a+2)}}{\cancel{(a+5)}\cancel{(a+1)}} \cdot \frac{\cancel{(a+1)}}{\cancel{(a+2)}\cancel{(a+2)}} \cdot \frac{\cancel{(a+2)}}{(a-1)} \quad \text{Rewrite}$$

$$\frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{(a-1)} \qquad \text{Multiply}$$

$$\frac{1}{(a-1)} \qquad \text{Reduced expression}$$

World Note

Indian mathematician *Aryabhata*, in the 6th century, published a work which included the rational expression $\frac{n(n+1)(n+2)}{6}$ for the sum of the first n squares $(1^1 + 2^2 + 3^2 + \ldots + n^2)$

8.2.4 Multiply and divide with rational functions

Example 8.13

Let $P(x) = \frac{4x^2+3x-1}{4x^2+9x+5}$ and $R(x) = \frac{x^2-2x-8}{4x^2+7x-2}$. Find and simplify $(P \cdot R)(x)$.

Solution.

First, we apply the definition for the product of two functions, then simplify.

$$(P \cdot R)(x) = P(x) \cdot R(x) \qquad \text{Replace } P(x) \text{ and } R(x)$$

$$(P \cdot R)(x) = \frac{4x^2+3x-1}{4x^2+9x+5} \cdot \frac{x^2-2x-8}{4x^2+7x-2} \qquad \text{Factor each numerator and denominator}$$

$$(P \cdot R)(x) = \frac{(4x-1)(x+1)}{(4x+5)(x+1)} \cdot \frac{(x-4)(x+2)}{(4x-1)(x+2)} \qquad \text{Reduce}$$

$$(P \cdot R)(x) = \frac{\cancel{(4x-1)}\cancel{(x+1)}}{(4x+5)\cancel{(x+1)}} \cdot \frac{(x-4)\cancel{(x+2)}}{\cancel{(4x-1)}\cancel{(x+2)}} \qquad \text{Rewrite the function}$$

$$(P \cdot R)(x) = \frac{x-4}{4x+5} \qquad \text{Product of } P \text{ and } R$$

Example 8.14

Let $P(x) = \frac{3x^2+14x+8}{3x^2+8x-16}$ and $R(x) = \frac{3x^2-4x-4}{x^2-3x+2}$. Find and simplify $(P \div R)(x)$.

Solution.

First, we apply the definition for the division of two functions, then simplify.

$$(P \div R)(x) = P(x) \div R(x) \qquad \text{Replace } P(x) \text{ and } R(x)$$

$$(P \div R)(x) = \frac{3x^2+14x+8}{3x^2+8x-16} \div \frac{3x^2-4x-4}{x^2-3x+2} \qquad \text{Form the reciprocal of } R \text{ and write as multiplication}$$

$$(P \div R)(x) = \frac{3x^2+14x+8}{3x^2+8x-16} \cdot \frac{x^2-3x+2}{3x^2-4x-4} \qquad \text{Factor each numerator and denominator}$$

$$(P \div R)(x) = \frac{(3x+2)(x+4)}{(3x-4)(x+4)} \cdot \frac{(x-2)(x-1)}{(3x+2)(x-2)} \qquad \text{Reduce}$$

$$(P \div R)(x) = \frac{\cancel{(3x+2)}\cancel{(x+4)}}{(3x-4)\cancel{(x+4)}} \cdot \frac{\cancel{(x-2)}(x-1)}{\cancel{(3x+2)}\cancel{(x-2)}} \qquad \text{Rewrite the function}$$

$$(P \div R)(x) = \frac{x-1}{3x-4} \qquad \text{Quotient of } P \text{ and } R$$

8.2.5 Multiply and Divide Rational Expressions Homework

Simplify each expression.

1. $\dfrac{8x^2}{9} \cdot \dfrac{9}{2}$

2. $\dfrac{9n}{2n} \cdot \dfrac{7}{5n}$

3. $\dfrac{5x^2}{4} \cdot \dfrac{6}{5}$

4. $\dfrac{7(m-6)}{m-6} \cdot \dfrac{5m(7m-5)}{7(7m-5)}$

5. $\dfrac{7r}{7r(r+10)} \div \dfrac{r-6}{(r-6)^2}$

6. $\dfrac{25n+25}{5} \cdot \dfrac{4}{30n+30}$

7. $\dfrac{x-10}{35x+21} \div \dfrac{7}{35x+21}$

8. $\dfrac{x^2-6x-7}{x+5} \cdot \dfrac{x+5}{x-7}$

9. $\dfrac{8k}{24k^2-40k} \div \dfrac{1}{15k-25}$

10. $(n-8) \cdot \dfrac{6}{10n-80}$

11. $\dfrac{4m+36}{m+9} \cdot \dfrac{m-5}{5m^2}$

12. $\dfrac{3x-6}{12x-24}(x+3)$

13. $\dfrac{b+2}{40b^2-24b}(5b-3)$

14. $\dfrac{n-7}{6n-12} \cdot \dfrac{12-6n}{n^2-13n+42}$

15. $\dfrac{27a+36}{9a+63} \div \dfrac{6a+8}{2}$

16. $\dfrac{x^2-12x+32}{x^2-6x-16} \cdot \dfrac{7x^2+14x}{7x^2+21x}$

17. $(10m^2+100m) \cdot \dfrac{18m^3-36m^2}{20m^2-40m}$

18. $\dfrac{7p^2+25p+12}{6p+48} \cdot \dfrac{3p-8}{21p^2-44p-32}$

19. $\dfrac{10b^2}{30b+20} \cdot \dfrac{30b+20}{2b^2+10b}$

20. $\dfrac{7r^2-53r-24}{7r+2} \div \dfrac{49r+21}{49r+14}$

21. $\dfrac{8x}{3x} \div \dfrac{4}{7}$

22. $\dfrac{9m}{5m^2} \cdot \dfrac{7}{2}$

23. $\dfrac{10p}{5} \div \dfrac{8}{10}$

24. $\dfrac{7}{10(n+3)} \div \dfrac{n-2}{(n+3)(n-2)}$

25. $\dfrac{6x(x+4)}{x-3} \cdot \dfrac{(x-3)(x-6)}{6x(x-6)}$

26. $\dfrac{9}{b^2-b-12} \div \dfrac{b-5}{b^2-b-12}$

27. $\dfrac{v-1}{4} \cdot \dfrac{4}{v^2-11v+10}$

28. $\dfrac{1}{a-6} \cdot \dfrac{8a+80}{8}$

29. $\dfrac{p-8}{p^2-12p+32} \div \dfrac{1}{p-10}$

30. $\dfrac{x^2-7x+10}{x-2} \cdot \dfrac{x+10}{x^2-x-20}$

31. $\dfrac{2r}{r+6} \div \dfrac{2r}{7r+42}$

32. $\dfrac{2n^2-12n-54}{n+7} \div (2n+6)$

33. $\dfrac{21v^2+16v-16}{3v+4} \div \dfrac{35v-20}{v-9}$

34. $\dfrac{x^2+11x+24}{6x^3+18x^2} \cdot \dfrac{6x^3+6x^2}{x^2+5x-24}$

35. $\dfrac{k-7}{k^2-k-12} \cdot \dfrac{7k^2-28k}{8k^2-56k}$

36. $\dfrac{9x^3+54x^2}{x^2+5x-14} \cdot \dfrac{x^2+5x-14}{10x^2}$

37. $\dfrac{n-7}{n^2-2n-35} \div \dfrac{9n+54}{10n+50}$

38. $\dfrac{7x^2-66x+80}{49x^2+7x-72} \div \dfrac{7x^2+39x-70}{49x^2+7x-72}$

39. $\dfrac{35n^2-12n-32}{49n^2-91n+40} \cdot \dfrac{7n^2+16n-15}{5n+4}$

40. $\dfrac{12x+24}{10x^2+34x+28} \cdot \dfrac{15x+21}{5}$

41. $\dfrac{x^2-1}{2x-4} \cdot \dfrac{x^2-4}{x^2-x-2} \div \dfrac{x^2+x-2}{3x-6}$

42. $\dfrac{x^2+3x+9}{x^2+x-12} \cdot \dfrac{x^2+2x-8}{x^3-27} \div \dfrac{x^2-4}{x^2-6x+9}$

43. $\dfrac{a^3+b^3}{a^2+3ab+2b^2} \cdot \dfrac{3a-6b}{3a^2-3ab+3b^2} \div \dfrac{a^2-4b^2}{a+2b}$

44. $\dfrac{x^2+3x-10}{x^2+6x+5} \cdot \dfrac{2x^2-x-3}{2x^2+x-6} \div \dfrac{8x+20}{6x+15}$

Perform the indicated operation and simplify.

45. Let $f(x) = \dfrac{5x^2+8x+3}{5x^2+7x+2}$ and $g(x) = \dfrac{x^2-4x+3}{5x^2-2x-3}$. Find and simplify $(P \cdot R)(x)$.

46. Let $f(x) = \dfrac{4x^2-21x+5}{4x^2-23x+15}$ and $g(x) = \dfrac{x^2+5x+6}{4x^2+11x-3}$. Find and simplify $(P \cdot R)(x)$.

47. Let $P(x) = \dfrac{3x^2-10x+8}{3x^2-4x-4}$ and $R(x) = \dfrac{3x^2+8x-16}{x^2+5x+4}$. Find and simplify $(P \div R)(x)$.

48. Let $P(x) = \dfrac{4x^2+19x-5}{4x^2+17x-15}$ and $R(x) = \dfrac{4x^2-21x+5}{x^2-3x-10}$. Find and simplify $(P \div R)(x)$.

8.3 Obtain the lowest common denominator

As with fractions in arithmetic, the least common denominator or LCD is the lowest common multiple (LCM) of the denominators. Since rational expressions are fractions with polynomials, we use the LCD to add and subtract rational expression with different denominators. In this section, we obtain LCDs of rational expressions. First, let's take a look at the method in finding the LCM in arithmetic.

8.3.1 Obtain the LCM in arithmetic

Example 8.15

Find LCM$(3, 6, 15)$.

Solution.

Find the prime factorization of each number in your set.

$$\begin{aligned} 3 &= 3 \\ 6 &= 2 \cdot 3 \\ 15 &= 3 \cdot 5 \end{aligned}$$

Next, take one of each factor and, for repeated factors, take the highest exponent. Hence, the LCM$(3, 6, 15) = 2 \cdot 3 \cdot 5 = 30$. Notice all factors of each number is in the LCM:

$$\underbrace{2 \cdot 3}_{6} \cdot 5$$

$$2 \cdot \underbrace{3 \cdot 5}_{15}$$

8.3.2 Obtain the LCM with monomials

We use the same method as in example 8.15, but now with variables.

Example 8.16

Find the LCM$(4x^2y^5, 6x^4y^3z^6)$.

Solution.

Find the prime factorization of each expression in your set.

$$\begin{aligned} 4x^2y^5 &= 2^2x^2y^5 \\ 6x^4y^3z^6 &= 2 \cdot 3 \cdot x^4y^3z^6 \end{aligned}$$

Next, take one of each factor and, for repeated factors, take the highest exponent. Hence, the LCM$(4x^2y^5, 6x^4y^3z^6) = 2^2 \cdot 3 \cdot x^4 \cdot y^5 \cdot z^6 = 12x^4y^3z^6$. Notice we take the highest exponent of repeated factors so that all factors are contained in the LCM.

8.3.3 Obtain the LCM with polynomials

We use the same method, but now we factor using factoring techniques to obtain the LCM between polynomials. Recall, all factors are contained in the LCM.

Example 8.17

Find the $\text{LCM}(x^2 + 2x - 3, x^2 - x - 12)$.

Solution.

Find the prime factorization of each expression in your set.

$$x^2 + 2x - 3 = (x + 3)(x - 1)$$
$$x^2 - x - 12 = (x - 4)(x + 3)$$

Next, take one of each factor and, for repeated factors, take the highest exponent. Hence, the $\text{LCM}(x^2 + 2x - 3, x^2 - x - 12) = (x - 1)(x + 3)(x - 4)$. Notice all factors are contained in the LCM:

$$\underbrace{(x - 1)(x + 3)}_{x^2+2x-3}(x - 4)$$
$$(x - 1)\underbrace{(x + 3)(x - 4)}_{x^2-x-12}$$

Example 8.18

Find the $\text{LCM}(x^2 - 10x + 25, x^2 - 14x + 45)$.

Solution.

Find the prime factorization of each expression in your set.

$$x^2 - 10x + 25 = (x - 5)^2$$
$$x^2 - 14x + 45 = (x - 5)(x - 9)$$

Next, take one of each factor and, for repeated factors, take the highest exponent. Hence, the $\text{LCM}(x^2 - 10x + 25, x^2 - 14x + 45) = (x - 5)^2(x - 9)$.

Once we obtain the LCM of polynomial expressions, then this LCM can be used as the LCD in given rational expressions. We can then rewrite each fraction with the LCD. Recall, the LCD is the LCM of all denominators in the expression.

8.3.4 Rewrite fractions with the lowest common denominator

Example 8.19

Find the LCD between $\dfrac{5a}{4b^3c}$ and $\dfrac{3c}{6a^2b}$. Rewrite each fraction with the LCD.

Solution.

If we need to obtain the LCD, then we can follow a series of steps.

Step 1. Find the LCD, i.e., the LCM between denominators. In this case, we need to find the $\text{LCM}(4b^3c, 6a^2b)$.

$$4b^3c = 2^2 \cdot b^3c$$
$$6a^2b = 2 \cdot 3 \cdot a^2b$$

We can see that the $\text{LCM}(4b^3c, 6a^2b) = 2^2 \cdot 3 \cdot a^2 \cdot b^3 \cdot c = 12a^2b^3c$. This is the LCD.

Step 2. Next, we rewrite each fraction with the LCD.

$\dfrac{5a}{4b^3c}$ Multiply the numerator and denominator by $3a^2$

$\dfrac{5a}{4b^3c} \cdot \dfrac{3a^2}{3a^2}$ Notice we get $12a^2b^3c$ in the denominator

$\dfrac{15a^3}{12a^2b^3c}$ The denominator is the LCD✓

$\dfrac{3c}{6a^2b}$ Multiply the numerator and denominator by $2b^2c$

$\dfrac{3c}{6a^2b} \cdot \dfrac{2b^2c}{2b^2c}$ Notice we get $12a^2b^3c$ in the denominator

$\dfrac{6b^2c^2}{12a^2b^3c}$ The denominator is the LCD✓

Hence, $\dfrac{5a}{4b^3c}$ and $\dfrac{3c}{6a^2b}$ can be written in the equivalent form with the LCD$= 12a^2b^3c$ as

$$\frac{15a^3}{12a^2b^3c} \quad \text{and} \quad \frac{6b^2c^2}{12a^2b^3c},$$

respectively.

Example 8.20

Find the LCD between $\dfrac{5x}{x^2-5x-6}$ and $\dfrac{x-2}{x^2+4x+3}$. Rewrite each fraction with the LCD.

Solution.

If we need to obtain the LCD, then we can follow a series of steps.

Step 1. Find the LCD, i.e., the LCM between denominators. In this case, we need to find the LCM(x^2-5x-6, x^2+4x+3).

$$x^2-5x-6 = (x+1)(x-6)$$
$$x^2+4x+3 = (x+3)(x+1)$$

We can see that the LCM$(x^2-5x-6, x^2+4x+3) = (x+3)(x+1)(x-6)$. This is the LCD.

Step 2. Next, we rewrite each fraction with the LCD.

$$\frac{5x}{x^2-5x-6} \quad \text{Factor the denominator}$$

$$\frac{5x}{(x+1)(x-6)} \quad \text{Multiply the numerator and denominator by } (x+3)$$

$$\frac{5x}{(x+1)(x-6)} \cdot \frac{(x+3)}{(x+3)} \quad \text{Notice we get the LCD in the denominator}$$

$$\frac{5x(x+3)}{(x+1)(x-6)(x+3)} \quad \text{The denominator is the LCD}\checkmark$$

$$\frac{x-2}{x^2+4x+3} \quad \text{Factor the denominator}$$

$$\frac{(x-2)}{(x+3)(x+1)} \quad \text{Multiply the numerator and denominator by } (x-6)$$

$$\frac{(x-2)}{(x+3)(x+1)} \cdot \frac{(x-6)}{(x-6)} \quad \text{Notice we get the LCD in the denominator}$$

$$\frac{(x-2)(x-6)}{(x+3)(x+1)(x-6)} \quad \text{The denominator is the LCD}\checkmark$$

Hence, $\dfrac{5x}{x^2-5x-6}$ and $\dfrac{x-2}{x^2+4x+3}$ can be written in the equivalent form with the LCD$=$ $(x+3)(x+1)(x-6)$ as

$$\frac{5x(x+3)}{(x+1)(x-6)(x+3)} \quad \text{and} \quad \frac{(x-2)(x-6)}{(x+3)(x+1)(x-6)},$$

respectively.

World Note

When the Egyptians began working with fractions, they expressed all fractions as a sum of a unit fraction. Rather than $\frac{4}{5}$, they would write the fraction as the sum, $\frac{1}{2}+\frac{1}{4}+\frac{1}{20}$. An interesting problem with this system is this is not a unique representation of $\frac{4}{5}$; $\frac{4}{5}$ is also equal to the sum $\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{10}$.

8.3.5 Obtain the Lowest Common Denominator Homework

Find the equivalent numerator.

1. $\dfrac{3}{8} = \dfrac{?}{48}$

2. $\dfrac{a}{x} = \dfrac{?}{xy}$

3. $\dfrac{2}{3a^3b^2c} = \dfrac{?}{9a^5b^2c^4}$

4. $\dfrac{2}{x+4} = \dfrac{?}{x^2-16}$

5. $\dfrac{x-4}{x+2} = \dfrac{?}{x^2+5x+6}$

6. $\dfrac{a}{5} = \dfrac{?}{5a}$

7. $\dfrac{5}{2x^2} = \dfrac{?}{8x^3y}$

8. $\dfrac{4}{3a^5b^2c^4} = \dfrac{?}{9a^5b^2c^4}$

9. $\dfrac{x+1}{x-3} = \dfrac{?}{x^2-6x+9}$

10. $\dfrac{x-6}{x+3} = \dfrac{?}{x^2-2x-15}$

Find the lowest common multiple.

11. $2a^3, 6a^4b^2, 4a^3b^5$

12. $x^2-3x, x-3, x$

13. $x+2, x-4$

14. $x^2-25, x+5$

15. x^2+3x+2, x^2+5x+6

16. $5x^2y, 25x^3y^5z$

17. $4x-8, x-2, 4$

18. $x, x-7, x+1$

19. x^2-9, x^2-6x+9

20. $x^2-7x+10, x^2-2x-15, x^2+x-6$

Find the LCD and rewrite each fraction with the LCD.

21. $\dfrac{3a}{5b^2}, \dfrac{2}{10a^3b}$

22. $\dfrac{x+2}{x-3}, \dfrac{x-3}{x+2}$

23. $\dfrac{x}{x^2-16}, \dfrac{3x}{x^2-8x+16}$

24. $\dfrac{x+1}{x^2-36}, \dfrac{2x+3}{x^2+12x+36}$

25. $\dfrac{4x}{x^2-x-6}, \dfrac{x+2}{x-3}$

26. $\dfrac{3x}{x-4}, \dfrac{2}{x+2}$

27. $\dfrac{5}{x^2-6x}, \dfrac{2}{x}, \dfrac{-3}{x-6}$

28. $\dfrac{5x+1}{x^2-3x-10}, \dfrac{4}{x-5}$

29. $\dfrac{3x+1}{x^2-x-12}, \dfrac{2x}{x^2+4x+3}$

30. $\dfrac{3x}{x^2-6x+8}, \dfrac{x-2}{x^2+x-20}, \dfrac{5}{x^2+3x-10}$

8.4 Add and subtract rational expressions

Adding and subtracting rational expressions are identical to adding and subtracting with numerical fractions. Recall, when adding with a common denominator, we add across numerators and keep the same denominator. This is the same method we use with rational expressions. Note, *methods never change, only problems.*

Helpful tips when adding and subtracting rational expressions

For adding and subtracting with rational expressions, here are some helpful tips:

- Identify the denominators: are they the same or different?
- Combine the rational expressions into one expression.
- Once combined into one expression, then reduce the fraction, if possible.
- A fraction is reducible only if there is a GCF between the numerator and denominator.
- If the numerator and denominator cannot be factored, they are unlikely to have any common factors.

8.4.1 Add or subtract rational expressions with a common denominator

Recall. We can use the same properties for adding or subtracting fractions with common denominators also for adding and subtracting rational expressions with common denominators:

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$$

Example 8.21

Add: $\dfrac{x-4}{x^2-2x-8} + \dfrac{x+8}{x^2-2x-8}$

Solution.

Using the helpful tips above, the denominators are the same. Let's combine into one fraction by adding across numerators and keeping the denominator the same:

$$\frac{x-4}{x^2-2x-8} + \frac{x+8}{x^2-2x-8} \quad \text{Like denominators, add across numerators}$$

$$\frac{2x+4}{x^2-2x-8} \quad \text{Factor the numerator and denominator}$$

$$\frac{2(x+2)}{(x+2)(x-4)} \quad \text{Reduce out a factor of } (x+2)$$

$$\frac{2\cancel{(x+2)}}{\cancel{(x+2)}(x-4)} \quad \text{Rewrite}$$

$$\frac{2}{(x-4)} \quad \text{Sum}$$

Notice, we had a GCF in the numerator. This is when we know the fraction may be reducible and we factor the GCF and determine whether the expression is reducible.

Subtraction with common denominators follows the same pattern. However, with subtraction, we first distribute the subtraction through the numerator. Then simplify as usual. This process is the same as "add the opposite" when subtracting with negative integers.

Example 8.22

Subtract: $\dfrac{6x-12}{3x-6}-\dfrac{15x-6}{3x-6}$

Solution.

Using the helpful tips above, the denominators are the same. Let's combine into one fraction by subtracting across numerators and keeping the denominator the same:

$$\frac{6x-12}{3x-6}-\frac{15x-6}{3x-6} \quad \text{Like denominators, subtract across numerators}$$

$$\frac{6x-12-(15x-6)}{3x-6} \quad \text{Simplify the numerator}$$

$$\frac{-9x-6}{3x-6} \quad \text{Factor the numerator and denominator}$$

$$\frac{-3(3x+2)}{3(x-2)} \quad \text{Reduce out a factor of 3}$$

$$\frac{-\cancel{3}(3x+2)}{\cancel{3}(x-2)} \quad \text{Rewrite}$$

$$\frac{-(3x+2)}{x-2} \quad \text{Difference}$$

Notice, we had a GCF in the numerator. This is when we know the fraction may be reducible and we factor the GCF and determine whether the expression is reducible.

World Note

The Rhind papyrus of Egypt from 1650 BC gives some of the earliest known symbols for addition and subtraction. For addition, a pair of legs walking in the direction one reads, and for subtraction, a pair of legs walking in the opposite direction.

8.4.2 Add and subtract rational expressions with unlike denominators

Recall. We can use the same properties for adding and subtracting integer fractions with unlike denominators for adding and subtracting rational expressions with unlike denominators.

Example 8.23

Add: $\dfrac{7a}{3a^2b}+\dfrac{4b}{6ab^4}$

Solution.

Using the helpful tips above, the denominators are different. We need to find the LCD, rewrite each fraction with the LCD, then combine into one fraction.

$\dfrac{7a}{3a^2b}+\dfrac{4b}{6ab^4}$	Unlike denominators; LCD $=6a^2b^4$
$\dfrac{2b^3}{2b^3}\cdot\dfrac{7a}{3a^2b}+\dfrac{4b}{6ab^4}\cdot\dfrac{a}{a}$	Rewrite each fraction with the LCD
$\dfrac{14ab^3}{6a^2b^4}+\dfrac{4ab}{6a^2b^4}$	Same denominator, add across numerators
$\dfrac{14ab^3+4ab}{6a^2b^4}$	Factor the numerator
$\dfrac{2ab(7b^3+2)}{6a^2b^4}$	Reduce out a factor of $2ab$
$\dfrac{\cancel{2ab}(7b^3+2)}{\cancelto{3}{6}\cancelto{1}{a^2}\cancelto{3}{b^4}}$	Rewrite
$\dfrac{7b^3+2}{3ab^3}$	Sum

Since there isn't a GCF in the numerator, as stated in the helpful tips, we cannot further reduce the fraction.

Example 8.24

Subtract: $\dfrac{4}{5a}-\dfrac{7b}{4a^2}$

Solution.

Using the helpful tips above, the denominators are different. We need to find the LCD, rewrite each fraction with the LCD, then combine into one fraction.

$\dfrac{4}{5a}-\dfrac{7b}{4a^2}$	Unlike denominators; LCD $=20a^2$
$\dfrac{4a}{4a}\cdot\dfrac{4}{5a}-\dfrac{7b}{4a^2}\cdot\dfrac{5}{5}$	Rewrite each fraction with the LCD
$\dfrac{16a}{20a^2}-\dfrac{35b}{20a^2}$	Same denominator, subtract across numerators
$\dfrac{16a-35b}{20a^2}$	Difference

Since there isn't a GCF in the numerator, as stated in the helpful tips, we cannot further reduce the fraction.

Example 8.25

Add: $\dfrac{6}{8a+4}+\dfrac{3a}{8}$

Solution.

Using the helpful tips above, the denominators are different. We need to find the LCD, rewrite each fraction with the LCD, then combine into one fraction. To determine the LCD, we have to factor the binomial in the first fraction's denominator.

$$\begin{array}{rl}
\dfrac{6}{8a+4}+\dfrac{3a}{8} & \text{Factor the first denominator} \\[2ex]
\dfrac{6}{4(2a+1)}+\dfrac{3a}{8} & \text{Unlike denominators; LCD} = 8(2a+1) \\[2ex]
\dfrac{2}{2}\cdot\dfrac{6}{4(2a+1)}+\dfrac{3a}{8}\cdot\dfrac{(2a+1)}{(2a+1)} & \text{Rewrite each fraction with the LCD} \\[2ex]
\dfrac{12}{8(2a+1)}+\dfrac{3a(2a+1)}{8(2a+1)} & \text{Same denominator, add across numerators} \\[2ex]
\dfrac{12+6a^2+3a}{8(2a+1)} & \text{Factor the numerator} \\[2ex]
\dfrac{3(2a^2+a+4)}{8(2a+1)} & \text{Expression is irreducible} \\[2ex]
\dfrac{3(2a^2+a+4)}{8(2a+1)} & \text{Sum}
\end{array}$$

Notice there is a GCF of 3 in the numerator, but 3 isn't a common factor. However, we still need to factor the GCF where possible from the numerator to verify that we can either reduce or not reduce the expression.

Example 8.26

Subtract: $\dfrac{x+1}{x-4}-\dfrac{x+1}{x^2-7x+12}$

Solution.

Using the helpful tips above, the denominators are different. We need to find the LCD, rewrite each fraction with the LCD, then combine into one fraction. To determine the LCD, we have to factor the trinomial in the second fraction's denominator.

$$\frac{x+1}{x-4} - \frac{x+1}{x^2-7x+12} \quad \text{Factor the second denominator}$$

$$\frac{x+1}{(x-4)} - \frac{x+1}{(x-3)(x-4)} \quad \text{Unlike denominators; LCD} = (x-3)(x-4)$$

$$\frac{(x-3)}{(x-3)} \cdot \frac{(x+1)}{(x-4)} - \frac{(x+1)}{(x-3)(x-4)} \quad \text{Rewrite each fraction with the LCD}$$

$$\frac{(x-3)(x+1)}{(x-3)(x-4)} - \frac{(x+1)}{(x-3)(x-4)} \quad \text{Same denominator, FOIL, subtract across numerators}$$

$$\frac{x^2-2x-3-(x-1)}{(x-3)(x-4)} \quad \text{Simplify the numerator}$$

$$\frac{x^2-3x-4}{(x-3)(x-4)} \quad \text{Factor the numerator}$$

$$\frac{(x-4)(x+1)}{(x-3)(x-4)} \quad \text{Reduce out a factor of } (x-4)$$

$$\frac{\cancel{(x-4)}(x+1)}{(x-3)\cancel{(x-4)}} \quad \text{Rewrite}$$

$$\frac{(x+1)}{(x-3)} \quad \text{Difference}$$

Recall, we do not reduce terms, only factors. Thus, the fraction above is the difference.

Warning!

We are not allowed to reduce terms, *only factors*.

8.4.3 Add and subtract with rational functions

Example 8.27

Let $P(x) = \dfrac{x+6}{x+5}$ and $R(x) = \dfrac{x+3}{x-9}$. Add and simplify $(P+R)(x)$.

Solution.

Using the helpful tips above, the denominators are different. We need to find the LCD, rewrite each fraction with the LCD, then combine into one fraction. To determine the LCD, we look at the denominators.

$$
\begin{aligned}
(P+R)(x) &= P(x)+R(x) && \text{Replace } P \text{ and } R\\
(P+R)(x) &= \frac{x+6}{x+5}+\frac{x+3}{x-9} && \text{Unlike denominators; LCD} = (x+5)(x-9)\\
(P+R)(x) &= \frac{(x-9)}{(x-9)}\cdot\frac{(x+6)}{(x+5)}+\frac{(x+3)}{(x-9)}\cdot\frac{(x+5)}{(x+5)} && \text{Rewrite each fraction with the LCD}\\
(P+R)(x) &= \frac{(x-9)(x+6)}{(x-9)(x+5)}+\frac{(x+3)(x+5)}{(x-9)(x+5)} && \text{Multiply each numerator}\\
(P+R)(x) &= \frac{x^2-3x-54}{(x-9)(x+5)}+\frac{x^2+8x+15}{(x-9)(x+5)} && \text{Same denominator, add across numerators}\\
(P+R)(x) &= \frac{2x^2+5x-39}{(x-9)(x+5)} && \text{Expression is irreducible}\\
(P+R)(x) &= \frac{2x^2+5x-39}{(x-9)(x+5)} && \text{Sum of } P \text{ and } R
\end{aligned}
$$

Since the numerator isn't factorable, as stated in the helpful tips, we cannot further reduce the fraction.

Example 8.28

Let $f(x)=\dfrac{x-3}{x+5}$ and $g(x)=\dfrac{-5x+7}{x^2+6x+5}$. Subtract and simplify $(f-g)(x)$.

Solution.

Using the helpful tips above, the denominators are different. We need to find the LCD, rewrite each fraction with the LCD, then combine into one fraction. To determine the LCD, we look at the denominators.

$$
\begin{aligned}
(f-g)(x) &= f(x)-g(x) && \text{Replace } f \text{ and } g\\
(f-g)(x) &= \frac{x-3}{x+5}-\frac{-5x+7}{x^2+6x+5} && \text{Factor the second denominator}\\
(f-g)(x) &= \frac{x-3}{x+5}-\frac{-5x+7}{(x+5)(x+1)} && \text{Unlike denominators; LCD} = (x+5)(x+1)\\
(f-g)(x) &= \frac{(x+1)}{(x+1)}\cdot\frac{(x-3)}{(x+5)}-\frac{-5x+7}{(x+5)(x+1)} && \text{Rewrite each fraction with the LCD}\\
(f-g)(x) &= \frac{(x+1)(x-3)}{(x+5)(x+1)}-\frac{-5x+7}{(x+5)(x+1)} && \text{Multiply the first numerator}\\
(f-g)(x) &= \frac{x^2-2x-3}{(x+5)(x+1)}-\frac{-5x+7}{(x+5)(x+1)} && \text{Same denominator, subtract across numerators}\\
(f-g)(x) &= \frac{x^2-2x-3-(-5x+7)}{(x+5)(x+1)} && \text{Simplify the numerator}
\end{aligned}
$$

$$(f-g)(x) = \frac{x^2+3x-10}{(x+5)(x+1)} \qquad \text{Factor the numerator}$$

$$(f-g)(x) = \frac{(x+5)(x-2)}{(x+5)(x+1)} \qquad \text{Reduce out a factor of } x+5$$

$$(f-g)(x) = \frac{\cancel{(x+5)}(x-2)}{\cancel{(x+5)}(x+1)} \qquad \text{Rewrite}$$

$$(f-g)(x) = \frac{(x-2)}{(x+1)} \qquad \text{Difference of } f \text{ and } g$$

8.4.4 Add and Subtract Rational Expressions Homework

Add or subtract the rational expressions. Simplify completely.

1. $\dfrac{2}{a+3}+\dfrac{4}{a+3}$

2. $\dfrac{t^2+4t}{t-1}+\dfrac{2t-7}{t-1}$

3. $\dfrac{2x^2+3}{x^2-6x+5}-\dfrac{x^2-5x+9}{x^2-6x+5}$

4. $\dfrac{5}{6r}-\dfrac{5}{8r}$

5. $\dfrac{8}{9t^3}+\dfrac{5}{6t^2}$

6. $\dfrac{a+2}{2}-\dfrac{a-4}{4}$

7. $\dfrac{x-1}{4x}-\dfrac{2x+3}{x}$

8. $\dfrac{5x+3y}{2x^2y}-\dfrac{3x+4y}{xy^2}$

9. $\dfrac{2z}{z-1}-\dfrac{3z}{z+1}$

10. $\dfrac{8}{x^2-4}-\dfrac{3}{x+2}$

11. $\dfrac{t}{t-3}-\dfrac{5}{4t-12}$

12. $\dfrac{2}{5x^2+5x}-\dfrac{4}{3x+3}$

13. $\dfrac{t}{y-t}-\dfrac{y}{y+t}$

14. $\dfrac{x}{x^2+5x+6}-\dfrac{2}{x^2+3x+2}$

15. $\dfrac{x}{x^2+15x+56}-\dfrac{7}{x^2+13x+42}$

16. $\dfrac{5x}{x^2-x-6}-\dfrac{18}{x^2-9}$

17. $\dfrac{2x}{x^2-1}-\dfrac{4}{x^2+2x-3}$

18. $\dfrac{x+1}{x^2-2x-35}+\dfrac{x+6}{x^2+7x+10}$

19. $\dfrac{4-a^2}{a^2-9}-\dfrac{a-2}{3-a}$

20. $\dfrac{2z}{1-2z}+\dfrac{3z}{2z+1}-\dfrac{3}{4z^2-1}$

21. $\dfrac{2x-3}{x^2+3x+2}+\dfrac{3x-1}{x^2+5x+6}$

22. $\dfrac{2x+7}{x^2-2x-3}-\dfrac{3x-2}{x^2+6x+5}$

23. $\dfrac{x^2}{x-2}-\dfrac{6x-8}{x-2}$

24. $\dfrac{a^2+3a}{a^2+5a-6}-\dfrac{4}{a^2+5a-6}$

25. $\dfrac{3}{x}+\dfrac{4}{x^2}$

26. $\dfrac{7}{xy^2}+\dfrac{3}{x^2y}$

27. $\dfrac{x+5}{8}+\dfrac{x-3}{12}$

28. $\dfrac{2a-1}{3a^2}+\dfrac{5a+1}{9a}$

29. $\dfrac{2c-d}{c^2d}-\dfrac{c+d}{cd^2}$

30. $\dfrac{2}{x-1}+\dfrac{2}{x+1}$

31. $\dfrac{2}{x-5}+\dfrac{3}{4x}$

32. $\dfrac{4x}{x^2-25}+\dfrac{x}{x+5}$

33. $\dfrac{2}{x+3}+\dfrac{4}{(x+3)^2}$

34. $\dfrac{3a}{4a-20}+\dfrac{9a}{6a-30}$

35. $\dfrac{x}{x-5}+\dfrac{x-5}{x}$

36. $\dfrac{2x}{x^2-1}-\dfrac{3}{x^2+5x+4}$

37. $\dfrac{2x}{x^2-9}+\dfrac{5}{x^2+x-6}$

38. $\dfrac{4x}{x^2-2x-3}-\dfrac{3}{x^2-5x+6}$

39. $\dfrac{x-1}{x^2+3x+2}+\dfrac{x+5}{x^2+4x+3}$

40. $\dfrac{3x+2}{3x+6}+\dfrac{x}{4-x^2}$

41. $\dfrac{4y}{y^2-1}-\dfrac{2}{y}-\dfrac{2}{y+1}$

42. $\dfrac{2r}{r^2-s^2}+\dfrac{1}{r+s}-\dfrac{1}{r-s}$

43. $\dfrac{x+2}{x^2-4x+3}+\dfrac{4x+5}{x^2+4x-5}$

44. $\dfrac{3x-8}{x^2+6x+8}+\dfrac{2x-3}{x^2+3x+2}$

Perform the indicated operation and simplify.

45. Let $P(x)=\dfrac{x}{x+6}$ and $R(x)=\dfrac{5x+6}{x^2+8x+12}$. Add and simplify $(P+R)(x)$.

46. Let $f(x)=\dfrac{x}{x+7}$ and $g(x)=\dfrac{10x+42}{x^2+10x+21}$. Add and simplify $(f+g)(x)$.

47. Let $S(x) = \dfrac{x-1}{x+4}$ and $V(x) = \dfrac{1x+14}{x^2+6x+8}$. Subtract and simplify $(S-V)(x)$.

48. Let $r(n) = \dfrac{n-2}{n-3}$ and $q(n) = \dfrac{11n-25}{n^2+2n-15}$. Subtract and simplify $(r-q)(n)$.

8.5 Compound rational expressions

Compound fractions have fractions in either the numerator, or denominator, or usually both, i.e., *fractions over fractions.* These expressions are simplified by one of two ways.

1. Simplify the numerator and denominator first, then divide as usual using techniques from this section.
2. Simplify by multiplying each term in the expression by the lowest common denominator. Then simplify as usual using techniques from the previous section.

Let's take a look at a few examples demonstrating both methods. We will start with a problem from arithmetic and then move on to algebraic expressions.

Example 8.29

Simplify $\dfrac{\frac{2}{3}-\frac{1}{4}}{\frac{5}{6}+\frac{1}{2}}$.

Solution.

We simplify the numerator and denominator first, then divide as usual using techniques from this section.

$$\frac{\frac{2}{3}-\frac{1}{4}}{\frac{5}{6}+\frac{1}{2}}$$ Rewrite each numerator and denominator in terms of the LCD

$$\frac{\frac{8}{12}-\frac{3}{12}}{\frac{5}{6}+\frac{3}{6}}$$ Simplify each numerator and denominator

$$\frac{\frac{5}{12}}{\frac{8}{6}}$$ Rewrite with the division sign

$$\frac{5}{12}\div\frac{8}{6}$$ Rewrite with multiplcation by reciprocating the second fraction

$$\frac{5}{12}\div\frac{6}{8}$$ Multiply across numerators and denominators

$$\frac{30}{96}$$ Reduce by a factor of 6

$$\frac{5}{16}$$ Result

Thus, the compound fraction reduces to $\dfrac{5}{16}$.

Example 8.30

Simplify $\dfrac{1-\frac{1}{x^2}}{1-\frac{1}{x}}$.

Solution.

We simplify by multiplying each term in the expression by the lowest common denominator. Then simplify as usual using techniques from the previous section.

$$\frac{1-\frac{1}{x^2}}{1-\frac{1}{x}} \qquad \text{Multiply each term by the LCD} = x^2$$

$$\frac{1\cdot x^2-\frac{1}{x^2}\cdot x^2}{1\cdot x^2-\frac{1}{x}\cdot x^2} \qquad \text{Simplify each term}$$

$$\frac{x^2-1}{x^2-x} \qquad \text{Factor the numerator and denominator}$$

$$\frac{(x+1)(x-1)}{x(x-1)} \qquad \text{Reduce by a factor of } (x-1)$$

$$\frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}} \qquad \text{Rewrite}$$

$$\frac{x+1}{x} \qquad \text{Result}$$

Thus, the compound fraction reduces to $\dfrac{x+1}{x}$.

Note

As best practice, we use the second method, where we multiply each term in the expression by the lowest common denominator because this technique reduced the fraction to one denominator and one numerator. The first method kept the compound fraction until we rewrote is as two expressions with division. Moving forward, students should always apply the second method, multiplying each term in the expression by the lowest common denominator.

Example 8.31

Simplify $\dfrac{\frac{3}{x+4}-2}{5+\frac{2}{x+4}}$

Solution.

We simplify by multiplying each term in the expression by the lowest common denominator. Then simplify as usual using techniques from the previous section.

$$\frac{\dfrac{3}{x+4}-2}{5+\dfrac{2}{x+4}} \quad \text{Multiply each term by the LCD } = (x+4)$$

$$\frac{\dfrac{3}{x+4}\cdot(x+4)-2\cdot(x+4)}{5\cdot(x+4)+\dfrac{2}{x+4}\cdot(x+4)} \quad \text{Simplify each term}$$

$$\frac{3-2(x+4)}{5(x+4)+2} \quad \text{Simplify}$$

$$\frac{3-2x-8}{5x+20+2} \quad \text{Combine like terms}$$

$$\frac{-2x-5}{5x+22} \quad \text{Result}$$

Thus, the compound fraction reduces to $\dfrac{-2x-5}{5x+22}$.

World Note

Sophie Germain is one of the most famous women in mathematics. Many prime numbers, which are important to finding an LCD, carry her name. *Germain primes* are prime numbers where one more than double the prime number is also prime. For example, 3 is prime and so is $2 \cdot 3 + 1 = 7$. The largest known Germain prime (at the time of printing) is $183027 \cdot 2^{265440} - 1$ which has 79,911 digits.

8.5.1 Compound Rational Expressions Homework

Simplify.

1. $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x^2}}$

2. $\dfrac{a-2}{\dfrac{4}{a}-a}$

3. $\dfrac{\dfrac{1}{a^2}-\dfrac{1}{a}}{\dfrac{1}{a^2}+\dfrac{1}{a}}$

4. $\dfrac{2-\dfrac{4}{x+2}}{5-\dfrac{10}{x+2}}$

5. $\dfrac{\dfrac{3}{2a-3}+2}{\dfrac{-6}{2a-3}-4}$

6. $\dfrac{\dfrac{x}{x+1}-\dfrac{1}{x}}{\dfrac{x}{x+1}+\dfrac{1}{x}}$

7. $\dfrac{\dfrac{3}{x}}{\dfrac{9}{x^2}}$

8. $\dfrac{\dfrac{a^2-b^2}{4a^2b}}{\dfrac{a+b}{16ab^2}}$

9. $\dfrac{1-\dfrac{3}{x}-\dfrac{10}{x^2}}{1+\dfrac{11}{x}+\dfrac{18}{x^2}}$

10. $\dfrac{1-\dfrac{2x}{3x-4}}{x-\dfrac{32}{3x-4}}$

11. $\dfrac{x-1+\dfrac{2}{x-4}}{x+3+\dfrac{6}{x-4}}$

12. $\dfrac{\dfrac{1}{y^2}-1}{1+\dfrac{1}{y}}$

13. $\dfrac{\dfrac{25}{a}-a}{5+a}$

14. $\dfrac{\dfrac{1}{b}+\dfrac{1}{2}}{\dfrac{4}{b^2-1}}$

15. $\dfrac{4+\dfrac{12}{2x-3}}{5+\dfrac{15}{2x-3}}$

16. $\dfrac{\dfrac{-5}{b-5}-3}{\dfrac{10}{b-5}+6}$

17. $\dfrac{\dfrac{2a}{a-1}-\dfrac{3}{a}}{\dfrac{-6}{a-1}-4}$

18. $\dfrac{\dfrac{x}{3x-2}}{\dfrac{x}{9x^2-4}}$

19. $\dfrac{1-\dfrac{1}{x}-\dfrac{6}{x^2}}{1-\dfrac{4}{x}+\dfrac{3}{x^2}}$

20. $\dfrac{\dfrac{15}{x^2}-\dfrac{2}{x}-1}{\dfrac{4}{x^2}-\dfrac{5}{x}+4}$

21. $\dfrac{1-\dfrac{12}{3x+10}}{x-\dfrac{8}{3x+10}}$

22. $\dfrac{x-5-\dfrac{18}{x+2}}{x+7+\dfrac{6}{x+2}}$

23. $\dfrac{x-4+\dfrac{9}{2x+3}}{x+3-\dfrac{5}{2x+3}}$

24. $\dfrac{\dfrac{2}{b}-\dfrac{5}{b+3}}{\dfrac{3}{b}+\dfrac{3}{b+3}}$

25. $\dfrac{\dfrac{2}{b^2}-\dfrac{5}{ab}-\dfrac{3}{a^2}}{\dfrac{2}{b^2}+\dfrac{7}{ab}+\dfrac{3}{a^2}}$

26. $\dfrac{\dfrac{y}{y+2}-\dfrac{y}{y-2}}{\dfrac{y}{y+2}+\dfrac{y}{y-2}}$

27. $\dfrac{\dfrac{1}{a}-\dfrac{3}{a-2}}{\dfrac{2}{a}+\dfrac{5}{a-2}}$

28. $\dfrac{\dfrac{1}{y^2}-\dfrac{1}{xy}-\dfrac{2}{x^2}}{\dfrac{1}{y^2}-\dfrac{3}{xy}+\dfrac{2}{x^2}}$

29. $\dfrac{\dfrac{x-1}{x+1}-\dfrac{x+1}{x-1}}{\dfrac{x-1}{x+1}+\dfrac{x+1}{x-1}}$

30. $\dfrac{\dfrac{x+1}{x-1}-\dfrac{1-x}{1+x}}{\dfrac{1}{(x+1)^2}+\dfrac{1}{(x-1)^2}}$

8.6 Rational Expressions: Answers to the Homework Exercises

Reduce Rational Expressions

1. 3

3. $\frac{1}{2}$

5. undefined; $a = -1$ is an excluded value

7. -10

9. 0

11. $-8, 4$

13. $0, -10$

15. $0, -\frac{1}{2}$

17. $\frac{3}{5a}$

19. $\frac{3m-4}{10}$

21. $\frac{1}{x+7}$

23. $\frac{n+6}{n-5}$

25. $\frac{2x-7}{5x-7}$

27. $\frac{2n-1}{9}$

29. $\frac{9x}{2}$

31. $\frac{1}{9}$

33. $\frac{7r+8}{8r}$

35. $\frac{3(x-3)}{5x+4}$

37. $\frac{9}{p+2}$

39. $\frac{2(m+2)}{5m-3}$

41. $\frac{7n-4}{4}$

43. $\frac{7a+9}{2(3a-2)}$

45. $\frac{5b-8}{5b+2}$

47. $\frac{7x-6}{(3x+4)(x+1)}$

Multiply and Divide Rational Expressions

1. $4x^2$

3. $\frac{3x^2}{2}$

5. $\frac{r-6}{r+10}$

7. $\frac{x-10}{7}$

9. 5

11. $\frac{4(m-5)}{5m^2}$

13. $\frac{b+2}{8b}$

15. $\frac{1}{a+7}$

17. $9m^2(m+10)$

19. $\frac{5b}{b+5}$

21. $\frac{14}{3}$

23. $\frac{5p}{2}$

25. $x+4$

27. $\frac{1}{v-10}$

29. $\frac{p-10}{p-4}$

31. 7

33. $\frac{v-9}{5}$

35. $\frac{7}{8(k+3)}$

37. $\frac{10}{9(n+6)}$

39. $n+3$

41. $\frac{3}{2}$

43. $\frac{1}{a+2b}$

45. $\frac{x-3}{5x+2}$

47. $\frac{x+1}{3x+2}$

Obtain the Lowest Common Denominator

1. 18

3. $6a^2c^3$

5. $x^2 - x - 12$

7. $20xy$

9. $x^2 - 2x - 3$

11. $12a^4b^5$

13. $(x+2)(x-4)$

15. $(x+1)(x+2)(x+3)$

17. $4(x-2)$

19. $(x-3)^2(x+3)$

21. $\frac{6a^4}{10a^3b^2}, \frac{2b}{10a^3b^2}$

23. $\frac{x^2-4x}{(x-4)^2(x+4)}, \frac{3x^2+12x}{(x-4)^2(x+4)}$

25. $\frac{4x}{(x-3)(x+2)}, \frac{x^2+4x+4}{(x-3)(x+2)}$

27. $\frac{5}{x(x-6)}, \frac{2x-12}{x(x-6)}, \frac{-3x}{x(x-6)}$

29. $\frac{3x^2+4x+1}{(x-4)(x+3)(x+1)}, \frac{2x^2-8x}{(x-4)(x+3)(x+1)}$

Add and Subtract Rational Expressions

1. $\frac{6}{a+3}$
3. $\frac{x+6}{x-5}$
5. $\frac{15t+16}{18t^3}$
7. $\frac{-7x-13}{4x}$
9. $\frac{-z^2+5z}{z^2-1}$
11. $\frac{4t-5}{4(t-3)}$
13. $\frac{t^2+2ty-y^2}{y^2-t^2}$
15. $\frac{x-8}{(x+8)(x+6)}$
17. $\frac{2x+4}{x^2+4x+3}$
19. $\frac{a-2}{a^2-9}$
21. $\frac{5(x-1)}{(x+1)(x+3)}$
23. $x-4$
25. $\frac{3x+4}{x^2}$
27. $\frac{5x+9}{24}$
29. $\frac{-c^2+cd-d^2}{c^2d^2}$
31. $\frac{11x+15}{4x(x+5)}$
33. $\frac{2x+10}{(x+3)^2}$
35. $\frac{2x^2-10x+25}{x(x-5)}$
37. $\frac{2x-5}{(x-3)(x-2)}$
39. $\frac{2x+7}{x^2+5x+6}$
41. $\frac{2}{y^2-y}$
43. $\frac{5x+5}{x^2+2x-15}$
45. $\frac{x+1}{x+2}$
47. $\frac{x-4}{x+2}$

Compound Rational Expressions

1. $\frac{x}{x-1}$
3. $-\frac{a-1}{a+1}$
5. $-\frac{1}{2}$
7. $\frac{x}{3}$
9. $\frac{x-5}{x+9}$
11. $\frac{x-2}{x+2}$
13. $\frac{5-a}{a}$
15. $\frac{4}{5}$
17. $\frac{2a^2-3a+3}{-4a^2-2a}$
19. $\frac{x+2}{x-1}$
21. $\frac{1}{x+4}$
23. $\frac{x-3}{x+4}$
25. $\frac{a-3b}{a+3b}$
27. $\frac{-2a-2}{3a-4}$
29. $-\frac{2x}{x^2+1}$

Chapter 9

Rational Equations and Applications

Chapter Objectives

By the end of this chapter, the student should be able to

- Identify extraneous values
- Apply methods of solving rational equations to solve rational equations
- Solve applications with rational equations including revenue, distance, and work-rate problems
- Identify graphs of rational functions

In this chapter, we'll solve rational equations and applications with rational equations by applying techniques from simplifying rational expressions. Additionally, we include discussion on rational inequalities and graphs of rational functions.

9.1 Rational equations

When solving rational equations, we can solve by using the same strategy we used to solve linear equations with fractions: *clearing denominators.* However, we first need to revisit excluded values.

9.1.1 Excluded values

Note

A rational expression is undefined where the denominator is zero. Recall, we cannot divide by zero, so it is critical we find these values and *exclude* them from the solution.

Example 9.1

Find the excluded value(s) of the expression: $\dfrac{-3z}{z+5}$

Solution.

Step 1. Set the denominator of the rational expression equal to zero:

$$z+5=0$$

Step 2. Solve the equation for z:

$$\begin{aligned} z + 5 &= 0 \\ z &= -5 \end{aligned}$$

Step 3. The values found in the previous step are the values excluded from the expression. Hence, the excluded value is $z = -5$.

Example 9.2

Find the excluded value(s) of the expression: $\dfrac{x^2-1}{3x^2+5x}$

Solution.

Step 1. Set the denominator of the rational expression equal to zero:

$$3x^2 + 5x = 0$$

Step 2. Solve the equation for x:

$$\begin{aligned} 3x^2 + 5x &= 0 \\ x(3x+5) &= 0 \\ x = 0 \quad &\text{or} \quad 3x + 5 = 0 \\ x = 0 \quad &\text{or} \quad 3x = -5 \\ x = 0 \quad &\text{or} \quad x = -\frac{5}{3} \end{aligned}$$

Step 3. The values found in the previous step are the values excluded from the expression. Hence, the excluded values are $x = 0$ and $x = -5$.

Definition

Recall, the excluded values are values in which make the expression undefined. Hence, when solving a rational equation, the solution(s) is any value(s) *except the excluded values*. If we obtain a solution that is an excluded value, we call this an ***extraneous solution***.

9.1.2 Clearing denominators using the LCD

Let's recall an example from solving linear equations with fractions. Let's be reminded of the process for clearing denominators when solving equations. In this section, we solve rational equations using the same process.

Example 9.3

Solve for x: $\dfrac{2}{3}x - \dfrac{5}{6} = \dfrac{3}{4}$

Solution.

This is a similar problem from solving linear equations with fractions. We will clear denominators by multiplying each term by the LCD.

$$\begin{aligned} \frac{2}{3}x - \frac{5}{6} &= \frac{3}{4} && \text{Multiply each term by LCD } = 12 \\ 12 \cdot \frac{2}{3}x - 12 \cdot \frac{5}{6} &= 12 \cdot \frac{3}{4} && \text{Clear denominators} \\ 8x - 10 &= 9 && \text{Isolate the variable term} \\ 8x &= 19 && \text{Solve for } x \\ x &= \frac{19}{8} && \text{Solution} \end{aligned}$$

Steps for solving rational equations

Step 1. Determine the excluded values of the equation.

Step 2. Clear denominators by multiplying each term by the lowest common denominator.

Step 3. Solve the equation.

Step 4. Verify that the solutions obtained are *not an excluded value.*

Example 9.4

Solve for x: $\dfrac{5x+5}{x+2} + 3x = \dfrac{x^2}{x+2}$

Solution.

We can solve by following the above steps.

Step 1. Determine the excluded values of the equation.

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

The excluded value is $x = -2$. This means we can obtain any solution *except for* $x = -2$.

Step 2. Clear denominators by multiplying each term by the lowest common denominator.

$$\begin{aligned} \frac{5x+5}{x+2} + 3x &= \frac{x^2}{x+2} && \text{Multiply each term by LCD } = (x+2) \\ (x+2) \cdot \frac{(5x+5)}{x+2} + (x+2) \cdot 3x &= (x+2) \cdot \frac{x^2}{x+2} && \text{Clear denominators} \\ 5x + 5 + 3x(x+2) &= x^2 \end{aligned}$$

Step 3. Solve the equation.

$$\begin{aligned}
5x+5+3x(x+2) &= x^2 && \text{Distribute}\\
5x+5+3x^2+6x &= x^2 && \text{Combine like terms}\\
3x^2+11x+5 &= x^2 && \text{Notice the term } x^2\text{; we solve by factoring}\\
2x^2+11x+5 &= 0 && \text{Zero on one side and factor the other side}\\
(2x+1)(x+5) &= 0 && \text{Apply the zero product rule}\\
2x+1=0 \text{ or } x+5 &= 0 && \text{Isolate variable terms}\\
2x=-1 \text{ or } x &= -5 && \text{Solve for } x\\
x=-\frac{1}{2} \text{ or } x &= -5 && \text{Solutions}
\end{aligned}$$

Step 4. Verify that the solutions obtained are *not an excluded value.* Since the excluded value is $x=-2$, and the solutions we obtained are $x=-\frac{1}{2}$ and $x=-5$, then we can conclude that $x=-\frac{1}{2}$ and $x=-5$ are, in fact, the solutions.

Example 9.5

Solve for x: $\dfrac{x}{x+2}+\dfrac{1}{x+1}=\dfrac{5}{(x+1)(x+2)}$

Solution.

We can solve by following the above steps.

Step 1. Determine the excluded values of the equation.

$$\begin{aligned}
x+2&=0 \qquad & x+1&=0\\
x&=-2 \qquad & x&=-1
\end{aligned}$$

The excluded values are $x=-2$ and $x=-1$. This means we can obtain any solution *except for* $x=-2$ and $x=-1$.

Step 2. Clear denominators by multiplying each term by the lowest common denominator.

$$\frac{x}{x+2}+\frac{1}{x+1}=\frac{5}{(x+1)(x+2)} \qquad \text{Multiply each term by LCD } =(x+2)(x+1)$$

Clear denominators:

$$(x+2)(x+1)\cdot\frac{x}{x+2}+(x+2)(x+1)\cdot\frac{1}{x+1}=(x+2)(x+1)\cdot\frac{5}{(x+1)(x+2)}$$

$$x(x+1)+1(x+2)=5$$

Step 3. Solve the equation.

$$\begin{aligned}
x(x+1)+1(x+2)&=5 && \text{Distribute}\\
x^2+x+x+2&=5 && \text{Combine like terms}\\
x^2+2x+2&=5 && \text{Notice the term } x^2\text{; we solve by factoring}\\
x^2+2x-3&=0 && \text{Zero on one side and factor the other side}\\
(x+3)(x-1)&=0 && \text{Apply the zero product rule}\\
x+3=0 \text{ or } x-1&=0 && \text{Isolate variable terms}\\
x=-3 \text{ or } x&=1 && \text{Solutions}
\end{aligned}$$

Step 4. Verify that the solutions obtained are *not an excluded value*. Since the excluded values are $x=-2$ and $x=-1$, and the solutions we obtained are $x=-3$ and $x=1$, then we can conclude that $x=-2$ and $x=-1$ are, in fact, the solutions.

9.1.3 Factoring denominators

In example 9.5, the denominators are factored, but this is not always the case. Often we will need to factor denominators before finding the LCD.

Example 9.6

Solve for t: $\dfrac{t}{t-1}-\dfrac{1}{t-2}=\dfrac{11}{t^2-3t+2}$

Solution.

We can solve by following the above steps.

Step 1. Determine the excluded values of the equation. Since we have three different denominators, we find excluded values for all different denominators.

$$\begin{aligned}
t-1&=0 & t-2&=0\\
t&=1 & t&=2
\end{aligned}
\qquad\qquad
\begin{aligned}
t^2-3t+2&=0\\
(t-2)(t-1)&=0\\
t-2=0 \qquad t-1&=0\\
t=2 \qquad t&=1
\end{aligned}$$

The excluded values are $t=1$ and $t=2$. This means we can obtain any solution *except for* $t=1$ and $t=2$. Even though we obtained repeated values, we still must find the excluded values for each denominator to verify the solution(s) in the last step.

Step 2. Clear denominators by multiplying each term by the lowest common denominator.

$$\begin{aligned}
\frac{t}{t-1}-\frac{1}{t-2}&=\frac{11}{t^2-3t+2} && \text{Factor denominator}\\
\frac{t}{t-1}-\frac{1}{t-2}&=\frac{11}{(t-2)(t-1)} && \text{Multiply each term by LCD } =(t-2)(t-1)
\end{aligned}$$

Clear denominators:

$$(t-2)(t-1)\cdot\frac{t}{t-1}-(t-2)(t-1)\cdot\frac{1}{t-2}=(t-2)(t-1)\cdot\frac{11}{(t-2)(t-1)}$$

$$t(t-2)-1(t-1)=11$$

Step 3. Solve the equation.

$$\begin{aligned} t(t-2)-1(t-1) &= 11 && \text{Distribute} \\ t^2-2t-t+1 &= 11 && \text{Combine like terms} \\ t^2-3t+1 &= 11 && \text{Notice the term } t^2\text{; we solve by factoring} \\ t^2-3t-10 &= 0 && \text{Zero on one side and factor the other side} \end{aligned}$$

$$\begin{aligned} (t+2)(t-5) &= 0 && \text{Apply the zero product rule} \\ t+2=0 \text{ or } t-5 &= 0 && \text{Isolate variable terms} \\ t=-2 \text{ or } t &= 5 && \text{Solutions} \end{aligned}$$

Step 4. Verify that the solutions obtained are *not an excluded value*. Since the excluded values are $t=1$ and $t=2$, and the solutions we obtained are $t=-2$ and $t=5$, then we can conclude that $t=-2$ and $t=5$ are, in fact, the solutions.

World Note

Rational functions are used to approximate or model more complex equations in science and engineering including physics, chemistry, biochemistry, optics and photography, and acoustics.

9.1.4 Solving rational equations with extraneous solutions

Example 9.7

Solve for n: $\dfrac{n}{n+5}-\dfrac{2}{n-9}=\dfrac{-11n+15}{n^2-4n-45}$

Solution.

We can solve by following the above steps.

Step 1. Determine the excluded values of the equation. Since $n^2-4n-45$ factors into $(n+5)(n-9)$, which are the factors of the denominators on the left side, we take factors $(n+5)$ and $(n-9)$ and find the excluded values.

$$\begin{aligned} n+5&=0 \qquad & n-9&=0 \\ n&=-5 & n&=9 \end{aligned}$$

The excluded values are $n=-5$ and $n=9$. This means we can obtain any solution *except for* $n=-5$ and $n=9$.

Step 2. Clear denominators by multiplying each term by the lowest common denominator.

$$\begin{aligned} \frac{n}{n+5}-\frac{2}{n-9} &= \frac{-11n+15}{n^2-4n-45} && \text{Factor denominator} \\ \frac{n}{n+5}-\frac{2}{n-9} &= \frac{-11n+15}{(n+5)(n-9)} && \text{Multiply each term by LCD } = (n+5)(n-9) \end{aligned}$$

Clear denominators:

$$(n+5)(n-9)\cdot\frac{n}{n+5}-(n+5)(n-9)\cdot\frac{2}{n-9}=(n+5)(n-9)\cdot\frac{-11n+15}{(n+5)(n-9)}$$

$$n(n-9)-2(n+5)=-11n+15$$

Step 3. Solve the equation.

$n(n-9)-2(n+5)=-11n+15$	Distribute
$n^2-9n-2n-10=-11n+15$	Combine like terms
$n^2-11n-10=-11n+15$	Notice the term n^2; we solve by factoring
$n^2-25=0$	Zero on one side and factor the other side
$(n+5)(n-5)=0$	Apply the zero product rule
$n+5=0$ or $n-5=0$	Isolate variable terms
$n=-5$ or $n=5$	Solutions

Step 4. Verify that the solutions obtained are *not an excluded value*. Since the excluded values are $n=-5$ and $n=9$, and the solutions we obtained are $n=-5$ and $n=5$, then $n=-5$ is an *extraneous solution* and we omit $n=-5$. Hence, we can conclude the solution is $n=5$.

9.1.5 Rational Equations Homework

Solve. Be sure to verify all solutions.

1. $3x - \frac{1}{2} - \frac{1}{x} = 0$

2. $x + \frac{20}{x-4} = \frac{5x}{x-4} - 2$

3. $x + \frac{6}{x-3} = \frac{2x}{x-3}$

4. $\frac{2x}{3x-4} = \frac{4x+5}{6x-1} - \frac{3}{3x-4}$

5. $\frac{3m}{2m-5} - \frac{7}{3m+1} = \frac{3}{2}$

6. $\frac{4-x}{1-x} = \frac{12}{3-x}$

7. $\frac{7}{y-3} - \frac{1}{2} = \frac{y-2}{y-4}$

8. $\frac{1}{x+2} - \frac{1}{2-x} = \frac{3x+8}{x^2-4}$

9. $\frac{x+1}{x-1} - \frac{x-1}{x+1} = \frac{5}{6}$

10. $\frac{3}{2x+1} + \frac{2x+1}{1-2x} = 1 - \frac{8x^2}{4x^2-1}$

11. $\frac{x-2}{x+3} - \frac{1}{x-2} = \frac{1}{x^2+x-6}$

12. $\frac{3}{x+2} + \frac{x-1}{x+5} = \frac{5x+20}{6x+24}$

13. $\frac{x}{x-1} - \frac{2}{x+1} = \frac{4x^2}{x^2-1}$

14. $\frac{2x}{x+1} - \frac{3}{x+5} = \frac{-8x^2}{x^2+6x+5}$

15. $\frac{x-5}{x-9} + \frac{x+3}{x-3} = \frac{-4x^2}{x^2-12x+27}$

16. $\frac{x-3}{x-6} + \frac{x+5}{x+3} = \frac{-2x^2}{x^2-3x-18}$

17. $\frac{4x+1}{x+3} + \frac{5x-3}{x-1} = \frac{8x^2}{x^2+2x-3}$

18. $\frac{6x+5}{2x^2-2x} - \frac{2}{1-x^2} = \frac{3x}{x^2-1}$

19. $x+1 = \frac{4}{x+1}$

20. $\frac{x^2+6}{x-1} + \frac{x-2}{x-1} = 2x$

21. $\frac{x-4}{x-1} = \frac{12}{3-x} + 1$

22. $\frac{4x}{2x-6} - \frac{4}{5x-15} = \frac{1}{2}$

23. $\frac{7}{3-x} + \frac{1}{2} = \frac{3}{4-x}$

24. $\frac{2}{3-x} - \frac{6}{8-x} = 1$

25. $\frac{x+2}{3x-1} - \frac{1}{x} = \frac{3x-3}{3x^2-x}$

26. $\frac{x-1}{x-3} + \frac{x+2}{x+3} = \frac{3}{4}$

27. $\frac{3x-5}{5x-5} + \frac{5x-1}{7x-7} - \frac{x-4}{1-x} = 2$

28. $\frac{x-1}{x-2} + \frac{x+4}{2x+1} = \frac{1}{2x^2-3x-2}$

29. $\frac{x}{x+3} - \frac{4}{x-2} = \frac{-5x^2}{x^2+x-6}$

30. $\frac{2x}{x+2} + \frac{2}{x-4} = \frac{3x}{x^2-2x-8}$

31. $\frac{x}{x+1} - \frac{3}{x+3} = \frac{-2x^2}{x^2+4x+3}$

32. $\frac{x-3}{x+6} + \frac{x-2}{x-3} = \frac{x^2}{x^2+3x-18}$

33. $\frac{x+3}{x-2} + \frac{x-2}{x+1} = \frac{9x^2}{x^2-x-2}$

34. $\frac{3x-1}{x+6} - \frac{2x-3}{x-3} = \frac{-3x^2}{x^2+3x-18}$

9.2 Rational Inequalities

In earlier chapters, we discussed many types of inequalities, and one of the most critical parts of inequalities is we know the solutions are always intervals of numbers. Hence, we now discuss rational inequalities where the solutions are intervals, but we take one critical detail into account. Since rational inequalities and equations contain excluded values, we must take this into consideration when obtaining the solution. In fact, we use the excluded values and x-intercepts, if any, to determine the intervals in which make the inequality true. Again, with inequalities, we are trying to find intervals of numbers in which the statements is true.

Steps for solving rational inequalities

Step 1. Rewrite the inequality so that only zero is on the right side.

Step 2. Determine where the rational expression is *undefined* or *equals zero*.

Step 3. Graph the values found in **Step 2.** on a number line into split into intervals.

Step 4. Take a test point within each interval and determine the sign of the result.

Step 5. Determine the solution, where the solution is the interval in which makes the inequality true.

Warning!

Be careful when determining the intervals. Recall, where the expression is undefined is not included in the interval, i.e., *always* use parenthesis for the values in which the expression is undefined.

Example 9.8

Solve $\dfrac{x-3}{x+1} > 0$.

Solution.

Step 1. Rewrite the inequality so that only zero is on the right side.

Since $\dfrac{x-3}{x+1} > 0$ already has zero on the right side, this step is done.

Step 2. Determine where the rational expression is *undefined* or *equals zero*.

To obtain where the rational expression **equals zero**, we set the numerator equal to zero:

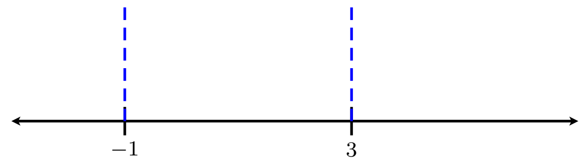
$$\begin{aligned} x-3&=0\\ x&=3 \end{aligned}$$

To obtain where the expression is **undefined**, we find its excluded value(s) by setting the denominator equal to zero:

$$\begin{aligned} x+1&=0\\ x&=-1 \end{aligned}$$

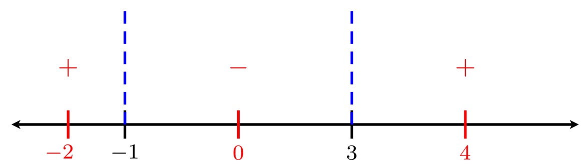
Step 3. Graph the values found in **Step 2.** on a number line into split into intervals.

Label -1 and 3 on a blank real number line:



Step 4. Take a test point within each interval and determine the sign of the result.

We take test values on each side of -1 and 3. Let's choose fairly easy numbers such as -2, 0, and 4. We plug-n-chug these numbers into $\dfrac{x-3}{x+1}$ and determine whether the value is positive or negative:



$$\text{letting } x = -2 \Longrightarrow \quad \frac{-2-3}{-2+1} = 5 > 0$$

$$\text{letting } x = 0 \Longrightarrow \quad \frac{0-3}{0+1} = -3 < 0$$

$$\text{letting } x = 4 \Longrightarrow \quad \frac{4-3}{4+1} = 0.2 > 0$$

Step 5. Determine the solution, where the solution is the interval in which makes the inequality true.

Since $\dfrac{x-3}{x+1} > 0$ (from **Step 1.**), then we are looking for where the test values are positive. Looking at the number line above, we see these are the values to the left of -1 and to the right of 3. Thus, the solution is $(-\infty, -1) \cup (3, \infty)$.

Note

We always use a **parenthesis** for the value that is **excluded** since this value makes the rational expression undefined. Using a bracket or parenthesis for where the expression equals zero is determined by the original inequality sign, e.g., $<, >$ use parenthesis, and $\geq, \leq$ use brackets.

Example 9.9

Solve $\dfrac{2x+3}{x-2} \leq 1$.

Solution.

Step 1. Rewrite the inequality so that only zero is on the right side.

We rewrite $\dfrac{2x+3}{x-2} \leq 1$ so that there is zero on the right side, and as one fraction:

$$\begin{aligned} \frac{2x+3}{x-2} &\leq 1 && \text{Subtract 1 from each side} \\ \frac{2x+3}{x-2} - 1 &\leq 0 && \text{Rewrite as one fraction where LCD: } (x-2) \\ \frac{2x+3}{x-2} - \frac{x-2}{x-2} &\leq 0 && \text{Subtract across numerators} \\ \frac{2x+3-x+2}{x-2} &\leq 0 && \text{Simplify} \\ \frac{x+5}{x-2} &\leq 0 && \text{We use this inequality to obtain the solution} \end{aligned}$$

Step 2. Determine where the rational expression is *undefined* or *equals zero.*

To obtain where the rational expression **equals zero**, we set the numerator equal to zero:

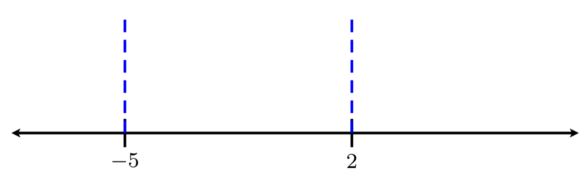
$$\begin{aligned} x+5 &= 0 \\ x &= -5 \end{aligned}$$

To obtain the **excluded** values, we set the denominator equal to zero:

$$\begin{aligned} x-2 &= 0 \\ x &= 2 \end{aligned}$$

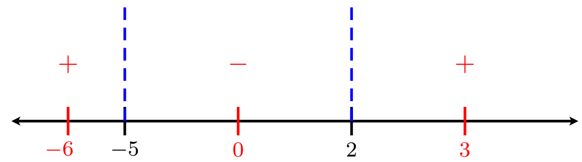
Step 3. Graph the values found in **Step 2.** on a number line into split into intervals.

Label -5 and 2 on a blank number line:



Step 4. Take a test point within each interval and determine the sign of the result.

We take test values on each side of -5 and 2. Let's choose fairly easy numbers such as -6, 0, and 3. We plug these numbers into $\dfrac{x+5}{x-2}$ and determine whether the value is positive or negative:



$$\text{letting } x = -6 \Longrightarrow \quad \frac{-6+5}{-6-2} = \frac{1}{8} > 0$$

$$\text{letting } x = 0 \Longrightarrow \quad \frac{0+5}{0-2} = -\frac{5}{2} < 0$$

$$\text{letting } x = 3 \Longrightarrow \quad \frac{3+5}{3-2} = 8 > 0$$

Step 5. Determine the solution, where the solution is the interval in which makes the inequality true.

Since $\dfrac{x+5}{x-2} \leq 0$ (from **Step 1.**), then we are looking for where the test values are negative or equal to zero. Looking at the number line above, we see these are the values between -5 and 2. Thus, the solution is $[-5, 2)$.

We used a bracket on -5 since the original inequality was $\leq$ and a parenthesis on 2 since 2 was an excluded value.

9.2.1 Rational Inequalities Homework

Solve the rational inequalities. Write the solution in interval notation.

1. $\dfrac{x-3}{x+1} < 0$

2. $\dfrac{x-7}{x+6} > 0$

3. $\dfrac{x-3}{x+1} < 1$

4. $\dfrac{x+27}{x+4} < 9$

5. $x + \dfrac{45}{x} < 14$

6. $\dfrac{(x-9)(x+9)}{x} \geq 0$

7. $\dfrac{(x+12)(x-4)}{x-1} \geq 0$

8. $\dfrac{3x}{7-x} < x$

9. $\dfrac{4x}{3-x} \geq 4x$

10. $\dfrac{8}{x-3} > \dfrac{6}{x-1}$

Find the values of x that satisfy the given condition for the function.

11. Solve $R(x) \geq 0$ if $R(x) = \dfrac{x+6}{x-4}$.

12. Solve $R(x) \leq 0$ if $R(x) = \dfrac{x-4}{x+8}$.

9.3 Work-rate problems

If it takes one person 4 hours to paint a room and another person 12 hours to paint the same room, working together they could paint the room even quicker. As it turns out, they would paint the room in 3 hours together. This is reasoned by the following logic. If the first person paints the room in 4 hours, she paints $\frac{1}{4}$ of the room each hour. If the second person takes 12 hours to paint the room, he paints $\frac{1}{12}$ of the room each hour. So together, each hour they paint $\frac{1}{4} + \frac{1}{12}$ of the room. Let's simplify this sum:

$$\frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$$

This means each hour, working together, they complete $\frac{1}{3}$ of the room. If $\frac{1}{3}$ of the room is painted each hour, it follows that it will take 3 hours to complete the entire room.

Work-rate equation

If the first person does a job in time A, a second person does a job in time B, and together they can do a job in time T (total). We can use the work-rate equation:

$$\underbrace{\frac{1}{A}}_{\text{job per time A}} + \underbrace{\frac{1}{B}}_{\text{job per time B}} = \underbrace{\frac{1}{T}}_{\text{job per time T}}$$

World Note

The Egyptians were the first to work with fractions. When the Egyptians wrote fractions, they were all unit fractions (a numerator of one). They used these types of fractions for about 2,000 years. Some believe that this cumbersome style of using fractions was used for so long out of tradition. Others believe the Egyptians had a way of thinking about and working with fractions that has been completely lost in history.

9.3.1 One unknown time

Example 9.10

Adam can clean a room in 3 hours. If his sister Maria helps, they can clean it in $2\frac{2}{5}$ hours. How long will it take Maria to do the job alone?

Solution.

We use the work-rate equation to model the problem, but before doing this, we can display the information on a table:

	time	**job per hour**
Adam	3	$1/3$
Maria	t	$1/t$
Together	$2\frac{2}{5}$	$1/2\frac{2}{5}$

Now, let's set up the equation and solve. Notice, $1/2\frac{2}{5}$ is an improper fraction and we can rewrite this as $1/\frac{12}{5} = 5/12$. We first clear denominators, then solve the linear equation as usual.

$$\frac{1}{3} + \frac{1}{t} = \frac{5}{12}$$

$$12t \cdot \frac{1}{3} + 12t \cdot \frac{1}{t} = 12t \cdot \frac{5}{12}$$

$$\begin{aligned} 4t + 12 &= 5t \\ 12 &= t \\ t &= 12 \end{aligned}$$

Thus, it would take Maria 12 hours to clean the room by herself.

Example 9.11

A sink can be filled by a pipe in 5 minutes, but it takes 7 minutes to drain a full sink. If both the pipe and the drain are open, how long will it take to fill the sink?

Solution.

We use the work-rate equation to model the problem, but before doing this, we can display the information on a table:

	time	**fill per minute**
Fill the sink	5	$1/5$
Drain the sink	7	$1/7$
Together	t	$1/t$

Now, let's set up the equation and solve. Notice, were are filling the sink and draining it. Since we are draining the sink, we are losing water as the sink fills. Hence, we will subtract the rate in which the sink drains. We first clear denominators, then solve the linear equation as usual.

$$\frac{1}{5} - \frac{1}{7} = \frac{1}{t}$$

$$35t \cdot \frac{1}{5} - 35t \cdot \frac{1}{7} = 35t \cdot \frac{1}{t}$$

$$\begin{aligned} 7t - 5t &= 35 \\ 2t &= 35 \\ t &= \frac{35}{2} \end{aligned}$$

Thus, it would take $\frac{35}{2}$ minutes to fill the sink, i.e., $17\frac{1}{2}$ minutes.

9.3.2 Two unknown times

Example 9.12

Mike takes twice as long as Rachel to complete a project. Together they can complete a project in 10 hours. How long will it take each of them to complete a project alone?

Solution.

We use the work-rate equation to model the problem, but before doing this, we can display the information on a table:

	time	**project per hour**
Mike	$2t$	$1/2t$
Rachel	t	$1/t$
Together	10	$1/10$

Now, let's set up the equation and solve. We first clear denominators, then solve the linear equation as usual.

$$\frac{1}{2t} + \frac{1}{t} = \frac{1}{10}$$

$$10t \cdot \frac{1}{2t} + 10t \cdot \frac{1}{t} = 10t \cdot \frac{1}{10}$$

$$\begin{aligned} 5 + 10 &= t \\ 15 &= t \\ t &= 15 \end{aligned}$$

Thus, it would take Rachel 15 hours to complete a project and Mike twice as long, 30 hours.

Example 9.13

Brittney can build a large shed in 10 days less than Cosmo. If they built it together, it would take them 12 days. How long would it take each of them working alone?

Solution.

We use the work-rate equation to model the problem, but before doing this, we can display the information on a table:

	time	**build per day**
Cosmo	t	$1/t$
Brittney	$t - 10$	$1/(t-10)$
Together	12	$1/12$

Now, let's set up the equation and solve. We first clear denominators, then solve the equation as usual.

$$\begin{aligned}
\frac{1}{t}+\frac{1}{t-10}&=\frac{1}{12} && \text{Apply the work-rate equation}\\
{\color{blue}12t(t-10)}\cdot\frac{1}{t}+{\color{blue}12t(t-10)}\cdot\frac{1}{t-10}&={\color{blue}12t(t-10)}\cdot\frac{1}{12} && \text{Clear denominators}\\
12(t-10)+12t&=t(t-10) && \text{Distribute}\\
12t-120+12t&=t^2-10t && \text{Combine like terms}\\
24t-120&=t^2-10t && \text{Notice the } t^2 \text{ term; solve by factoring}\\
t^2-34t+120&=0 && \text{Factor}\\
(t-4)(t-30)&=0 && \text{Apply zero product rule}\\
t-4=0 \text{ or } t-30&=0 && \text{Isolate variable terms}\\
t=4 \text{ or } t&=30 && \text{Solutions}
\end{aligned}$$

We obtained $t = 4$ and $t = 30$ for the solutions. However, we need to verify these solutions with Cosmo and Brittney's times. If $t = 4$, then Brittney's time would be $4 - 10 = -6$ days. This makes no sense since days are always positive. Thus, it would take Cosmo 30 days to build a shed and Brittney 10 less days, 20 days.

Example 9.14

An electrician can complete a job in one hour less than his apprentice. Together they do the job in 1 hour and 12 minutes. How long would it take each of them working alone?

Solution.

We use the work-rate equation to model the problem, but before doing this, we can display the information on a table. Notice the time given doing the job together: 1 hour and 12 minutes. Unfortunately, we cannot use this format in the work-rate equation. Hence, we need to convert this to the same time units: 1 hour and 12 minutes= $1\frac{12}{60}$ hours = 1.2 hours = $\frac{6}{5}$ hours.

	time	job per hour
Electrician	$t-1$	$1/(t-1)$
Apprentice	t	$1/t$
Together	$6/5$	$5/6$

Note, $\frac{1}{6/5}=\frac{5}{6}$. Now, let's set up the equation and solve. We first clear denominators, then solve the equation as usual.

$$\begin{aligned}
\frac{1}{t-1}+\frac{1}{t}&=\frac{5}{6} && \text{Apply the work-rate equation}\\
{\color{blue}6t(t-1)}\cdot\frac{1}{t-1}+{\color{blue}6t(t-1)}\cdot\frac{1}{t}&={\color{blue}6t(t-1)}\cdot\frac{5}{6} && \text{Clear denominators}\\
6t+6(t-1)&=5t(t-1) && \text{Distribute}\\
6t+6t-6&=5t^2-5t && \text{Combine like terms}\\
12t-6&=5t^2-5t && \text{Notice the } 5t^2 \text{ term; solve by factoring}
\end{aligned}$$

$$\begin{aligned}
5t^2 - 17t + 6 &= 0 && \text{Factor}\\
(5t-2)(t-3) &= 0 && \text{Apply zero product rule}\\
5t - 2 = 0 \text{ or } t - 3 &= 0 && \text{Isolate variable terms}\\
t = \frac{2}{5} \text{ or } t &= 3 && \text{Solutions}
\end{aligned}$$

We obtained $t = \frac{2}{5}$ and $t = 3$ for the solutions. However, we need to verify these solutions with the electrician and apprentice's times. If $t = \frac{2}{5}$, then the electrician's time would be $\frac{2}{5} - 1 = -\frac{3}{5}$ hours. This makes no sense since hours are always positive. Thus, it would take the apprentice 3 hours to complete a job and the electrician 1 less hour, 2 hours.

9.3.3 Work-rate Problems Homework

1. Bill's father can paint a room in two hours less than Bill can paint it. Working together they can complete the job in two hours and 24 minutes. How much time would each require working alone?

2. Of two inlet pipes, the smaller pipe takes four hours longer than the larger pipe to fill a pool. When both pipes are open, the pool is filled in three hours and forty-five minutes. If only the larger pipe is open, how many hours are required to fill the pool?

3. Jack can wash and wax the family car in one hour less than Bob can. The two working together can complete the job in $1\frac{1}{5}$ hours. How much time would each require if they worked alone?

4. If A can do a piece of work alone in 6 days and B can do it alone in 4 days, how long will it take the two working together to complete the job?

5. Working alone it takes John 8 hours longer than Carlos to do a job. Working together they can do the job in 3 hours. How long will it take each to do the job working alone?

6. A can do a piece of work in 3 days, B in 4 days, and C in 5 days each working alone. How long will it take them to do it working together?

7. A can do a piece of work in 4 days and B can do it in half the time. How long will it take them to do the work together?

8. A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take both pipes together to fill the tank?

9. If A can do a piece of work in 24 days and A and B together can do it in 6 days, how long would it take B to do the work alone?

10. A carpenter and his assistant can do a piece of work in $3\frac{3}{4}$ days. If the carpenter himself could do the work alone in 5 days, how long would the assistant take to do the work alone?

11. If Sam can do a certain job in 3 days, while it takes Fred 6 days to do the same job, how long will it take them, working together, to complete the job?

12. Tim can finish a certain job in 10 hours. It take his wife JoAnn only 8 hours to do the same job. If they work together, how long will it take them to complete the job?

13. Two people working together can complete a job in 6 hours. If one of them works twice as fast as the other, how long would it take the faster person, working alone, to do the job?

14. If two people working together can do a job in 3 hours, how long will it take the slower person to do the same job if one of them is 3 times as fast as the other?

15. A water tank can be filled by an inlet pipe in 8 hours. It takes twice that long for the outlet pipe to empty the tank. How long will it take to fill the tank if both pipes are open?

16. A sink can be filled from the faucet in 5 minutes. It takes only 3 minutes to empty the sink when the drain is open. If the sink is full and both the faucet and the drain are open, how long will it take to empty the sink?

17. It takes 10 hours to fill a pool with the inlet pipe. It can be emptied in 15 hrs with the outlet pipe. If the pool is half full to begin with, how long will it take to fill it from there if both pipes are open?

18. A sink is $\frac{1}{4}$ full when both the faucet and the drain are opened. The faucet alone can fill the sink in 6 minutes, while it takes 8 minutes to empty it with the drain. How long will it take to fill the remaining $\frac{3}{4}$ of the sink?

19. A sink has two faucets, one for hot water and one for cold water. The sink can be filled by a cold-water faucet in 3.5 minutes. If both faucets are open, the sink is filled in 2.1 minutes. How long does it take to fill the sink with just the hot-water faucet open?

20. A water tank is being filled by two inlet pipes. Pipe A can fill the tank in $4\frac{1}{2}$ hrs, while both pipes together can fill the tank in 2 hours. How long does it take to fill the tank using only pipe B?

21. A tank can be emptied by any one of three caps. The first can empty the tank in 20 minutes while the second takes 32 minutes. If all three working together could empty the tank in $8\frac{8}{59}$ minutes, how long would the third take to empty the tank?

22. One pipe can fill a cistern in $1\frac{1}{2}$ hours while a second pipe can fill it in $2\frac{1}{3}$ hrs. Three pipes working together fill the cistern in 42 minutes. How long would it take the third pipe alone to fill the tank?

23. Sam takes 6 hours longer than Susan to wax a floor. Working together they can wax the floor in 4 hours. How long will it take each of them working alone to wax the floor?

24. It takes Robert 9 hours longer than Paul to rapair a transmission. If it takes them $2\frac{2}{5}$ hours to do the job if they work together, how long will it take each of them working alone?

25. It takes Sally $10\frac{1}{2}$ minutes longer than Patricia to clean up their dorm room. If they work together they can clean it in 5 minutes. How long will it take each of them if they work alone?

26. A takes $7\frac{1}{2}$ minutes longer than B to do a job. Working together they can do the job in 9 minutes. How long does it take each working alone?

27. Secretary A takes 6 minutes longer than Secretary B to type 10 pages of manuscript. If they divide the job and work together it will take them $8\frac{3}{4}$ minutes to type 10 pages. How long will it take each working alone to type the 10 pages?

28. It takes John 24 minutes longer than Sally to mow the lawn. If they work together they can mow the lawn in 9 minutes. How long will it take each to mow the lawn if they work alone?

9.4 Uniform motion problems

We can recall uniform motion problems in the word problems chapter. We used the formula $r \cdot t = d$ and organized the given information in a table. Now, we use the equation as

$$t = \frac{d}{r}$$

We apply the same method in this section only the equations will be rational equations.

9.4.1 Uniform motion problems

Example 9.15

Greg went to a conference in a city 120 miles away. On the way back, due to road construction he had to drive 10 mph slower which resulted in the return trip taking 2 hours longer. How fast did he drive on the way to the conference?

Solution.

First, we can make a table to organize the given information and then create an equation. Let r represent the rate in which he drove to the conference.

	rate	time	distance
To the conference	r	t	120
From the conference	$r-10$	$t+2$	120

Now we can set up each equation.

$$t_{\text{to}} = \frac{120}{r} \qquad t_{\text{from}} + 2 = \frac{120}{r-10}$$

$$t_{\text{to}} = \frac{120}{r} \qquad t_{\text{from}} = \frac{120}{r-10} - 2$$

Since we solved for t in each equation, we can set the t's equal to each other and solve for r:

$$\begin{aligned}
t_{\text{to}} &= t_{\text{from}} && \text{Set } t\text{'s equal to each other}\\
\frac{120}{r} &= \frac{120}{r-10} - 2 && \text{Multiply by the LCD}\\
r(r-10)\cdot\frac{120}{r} &= r(r-10)\cdot\frac{120}{r-10} - r(r-10)\cdot 2 && \text{Clear denominators}\\
120(r-10) &= 120r - 2r(r-10) && \text{Distribute}\\
120r - 1200 &= 120r - 2r^2 + 20r && \text{Combine like terms}\\
120r - 1200 &= 140r - 2r^2 && \text{Notice the } 2r^2 \text{ term; solve by factoring}\\
2r^2 - 20r - 1200 &= 0 && \text{Reduce all terms by a factor of 2}\\
r^2 - 10r - 600 &= 0 && \text{Factor}\\
(r+20)(r-30) &= 0 && \text{Apply zero product rule}\\
r+20 = 0 \text{ or } r-30 &= 0 && \text{Isolate variable terms}\\
r = -20 \text{ or } r &= 30 && \text{Solutions}
\end{aligned}$$

Since the rate of the car is always positive, we omit the solution $r = -20$. Thus, Greg drove at a rate of 30 miles per hour to the conference.

World Note

The world's fastest man (at the time of printing) is Jamaican Usain Bolt who set the record of running 100 meters in 9.58 seconds on August 16, 2009 in Berlin. That is a speed of over 23 miles per hour.

9.4.2 Uniform motion problems with streams and winds

Another type of uniform motion problem is where a boat is traveling in a river with the current or against the current (or an airplane flying with the wind or against the wind). If a boat is traveling downstream, the current will push it or increase the rate by the speed of the current. If a boat is traveling upstream, the current will pull against it or decrease the rate by the speed of the current.

Example 9.16

A man rows down stream for 30 miles then turns around and returns to his original location, the total trip took 8 hours. If the current flows at 2 miles per hour, how fast would the man row in still water?

Solution.

First, we can make a table to organize the given information and then create an equation. Let r represent the rate in which the man would row in still water.

	rate	**time**	**distance**
Downstream	$r+2$	t	30
Upstream	$r-2$	$8-t$	30

Now we can set up each equation.

$$t_{\text{ds}} = \frac{30}{r+2} \qquad 8 - t_{\text{us}} = \frac{30}{r-2}$$

$$t_{\text{ds}} = \frac{30}{r+2} \qquad t_{\text{us}} = 8 - \frac{30}{r-2}$$

Since we solved for t in each equation, we can set the t's equal to each other and solve for r:

$$t_{\text{ds}} = t_{\text{us}} \qquad \text{Set } t\text{'s equal to each other}$$

$$\frac{30}{r+2} = 8 - \frac{30}{r-2} \qquad \text{Multiply by the LCD}$$

$$(r+2)(r-2)\cdot\frac{30}{r+2} = (r+2)(r-2)\cdot 8 - (r+2)(r-2)\cdot\frac{30}{r-2} \qquad \text{Clear denominators}$$

$$30(r-2) = 8(r+2)(r-2) - 30(r+2) \qquad \text{Distribute}$$

$$30r - 60 = 8r^2 - 32 - 30r - 60 \qquad \text{Combine like terms}$$

$$\begin{aligned}
30r - 60 &= 8r^2 - 30r - 92 && \text{Notice the } 8r^2 \text{ term; solve by factoring}\\
8r^2 - 60r - 32 &= 0 && \text{Reduce all terms by a factor of 4}\\
2r^2 - 15r - 8 &= 0 && \text{Factor}\\
(2r+1)(r-8) &= 0 && \text{Apply zero product rule}\\
2r + 1 = 0 \text{ or } r - 8 &= 0 && \text{Isolate variable terms}\\
r = -\frac{1}{2} \text{ or } r &= 8 && \text{Solutions}
\end{aligned}$$

Since the rate of the boat is always positive, we omit the solution $r = -\frac{1}{2}$. Thus, the man rowed at a rate of 8 miles per hour in still water.

9.4.3 Uniform Motion Problems Homework

1. A train traveled 240 kilometers at a certain speed. When the engine was replaced by an improved model, the speed was increased by 20 km/hr and the travel time for the trip was decreased by 1 hour. What was the rate of each engine?

2. The rate of the current in a stream is 3 km/hr. A man rowed upstream for 3 kilometers and then returned. The round trip required 1 hour and 20 minutes. How fast was he rowing?

3. A pilot flying at a constant rate against a headwind of 50 km/hr flew for 750 kilometers, then reversed direction and returned to his starting point. He completed the round trip in 8 hours. What was the speed of the plane?

4. Two drivers are testing the same model car at speeds that differ by 20 km/hr. The one driving at the slower rate drives 70 kilometers down a speedway and returns by the same route. The one driving at the faster rate drives 76 kilometers down the speedway and returns by the same route. Both drivers leave at the same time, and the faster car returns $\frac{1}{2}$ hour earlier than the slower car. At what rates were the cars driven?

5. An athlete plans to row upstream a distance of 2 kilometers and then return to his starting point in a total time of 2 hours and 20 minutes. If the rate of the current is 2 km/hr, how fast should he row?

6. An automobile goes to a place 72 miles away and then returns, the round trip occupying 9 hours. His speed in returning is 12 miles per hour faster than his speed in going. Find the rate of speed in both going and returning.

7. An automobile made a trip of 120 miles and then returned, the round trip occupying 7 hours. Returning, the rate was increased 10 miles an hour. Find the rate of each.

8. The rate of a stream is 3 miles an hour. If a crew rows downstream for a distance of 8 miles and then back again, the round trip occupying 5 hours, what is the rate of the crew in still water?

9. The railroad distance between two towns is 240 miles. If the speed of a train were increased 4 miles an hour, the trip would take 40 minutes less. What is the usual rate of the train?

10. By going 15 miles per hour faster, a train would have required 1 hour less to travel 180 miles. How fast did it travel?

11. Mr. Jones visits his grandmother who lives 100 miles away on a regular basis. Recently a new freeway has opend up and, although the freeway route is 120 miles, he can drive 20 mph faster on average and takes 30 minutes less time to make the trip. What is Mr. Jones' rate on both the old route and on the freeway?

12. If a train had traveled 5 miles an hour faster, it would have needed $1\frac{1}{2}$ hours less time to travel 150 miles. Find the rate of the train.

13. A traveler having 18 miles to go, calculates that his usual rate would make him one-half hour late for an appointment; he finds that in order to arrive on time he must travel at a rate one-half mile an hour faster. What is his usual rate?

9.5 Revenue problems

Revenue problems are problems where a person buys a certain number of items for a certain price per item. If we multiply the number of items by the price per item we will get the total value. We can recall revenue problems in the word problems chapter. We used the formula AVT:

$$\text{Amount} \cdot \text{Value} = \text{Total}$$

We will continue to use the same formula, but rewrite it to model with rational equations as

$$\text{Value} = \frac{\text{Total}}{\text{Amount}}$$

Example 9.17

A man buys several fish for \$56. After three fish die, he decides to sell the rest at a profit of \$5 per fish. His total profit was \$4. How many fish did he buy to begin with?

Solution.

First, we can make a table to organize the given information and then create an equation. Let n represent the number of fish and p be the price of each fish.

	Amount	Price	Total value
Buy	n	p	\$56
Sell	$n-3$	$p+5$	\$56 + \$4

Let's discuss the table for a moment. When the man purchased the fish, the total value of fish purchased was \$56. Since the price of each fish and the quantity purchased is unknown, we leave it as p and n, respectively. The man wants to sell the fish, but three died; hence, the amount left to sell is $n-3$. Since he wants to profit \$5 per fish, then we take the price the man bought the fish for and add \$5, $p+5$. It is given that his total profit was \$4, so his total value from selling the fish was the original value, \$56, plus the \$4 profit; hence, a total of \$60.

Finally, let's set up the equations and solve:

$$p_{\text{Buy}} = \frac{56}{n} \qquad p_{\text{Sell}} + 5 = \frac{60}{(n-3)}$$

$$p_{\text{Buy}} = \frac{56}{n} \qquad p_{\text{Sell}} = \frac{60}{(n-3)} - 5$$

Since we solved for p in each equation, we can set the p's equal to each other:

$$p_{\text{Buy}} = p_{\text{Sell}} \qquad \text{Set } p\text{'s equal to each other}$$

$$\frac{56}{n} = \frac{60}{(n-3)} - 5 \qquad \text{Multiply by the LCD}$$

$$n(n-3) \cdot \frac{56}{n} = n(n-3) \cdot \frac{60}{(n-3)} - n(n-3) \cdot 5 \qquad \text{Clear denominators}$$

$$56(n-3) = 60n - 5n(n-3) \qquad \text{Distribute}$$

$$
\begin{aligned}
56n - 168 &= 60n - 5n^2 + 15n && \text{Combine like terms}\\
56n - 168 &= 75n - 5n^2 && \text{Notice the } 5n^2 \text{ term; solve by factoring}\\
5n^2 - 19n - 168 &= 0 && \text{Factor}\\
(5n + 21)(n - 8) &= 0 && \text{Apply zero product rule}\\
5n + 21 = 0 \text{ or } n - 8 &= 0 && \text{Isolate variable terms}\\
n = -\frac{21}{5} \text{ or } n &= 8 && \text{Solutions}
\end{aligned}
$$

Since the quantity of fish is always positive, we omit the solution $n = -\frac{21}{5}$. Thus, the man purchased 8 fish.

Example 9.18

A group of students bought a couch for their dorm that cost \$96. However, 2 students failed to pay their share, so each student had to pay \$4 more. How many students were in the original group?

Solution.

First, we can make a table to organize the given information and then create an equation. Let n represent the number of students and p be the price of each share.

	Amount	**Price**	**Total value**
Original deal	n	p	\$96
Actual deal	$n - 2$	$p + 4$	\$96

Let's discuss the table for a moment. The original deal was every student in the original group with n number of students were going to split the total value of the couch valued at \$96. Since the price of each share and the number of students is unknown, we leave it as p and n, respectively. When it came to actually paying for the couch, 2 students didn't pay their share; hence, the number of students left to pay is $n - 2$. Since this increases each share from the rest of the group, then we take the original share and add \$4, $p + 4$.

Finally, let's set up the equations and solve:

$$
p_{\mathrm{O}} = \frac{96}{n} \qquad p_{\mathrm{A}} + 4 = \frac{96}{(n-2)}
$$

$$
p_{\mathrm{O}} = \frac{96}{n} \qquad p_{\mathrm{A}} = \frac{96}{(n-2)} - 4
$$

Since we solved for p in each equation, we can set the p's equal to each other:

$$
\begin{aligned}
p_{\mathrm{O}} &= p_{\mathrm{A}} && \text{Set } p\text{'s equal to each other}\\
\frac{96}{n} &= \frac{96}{(n-2)} - 4 && \text{Multiply by the LCD}\\
{\color{blue}n(n-2)} \cdot \frac{96}{n} &= {\color{blue}n(n-2)} \cdot \frac{96}{(n-2)} - {\color{blue}n(n-2)} \cdot 4 && \text{Clear denominators}\\
96(n-2) &= 96n - 4n(n-2) && \text{Distribute}
\end{aligned}
$$

$$
\begin{aligned}
96n - 192 &= 96n - 4n^2 + 8n && \text{Combine like terms} \\
96n - 192 &= 104n - 4n^2 && \text{Notice the } 4n^2 \text{ term; solve by factoring} \\
4n^2 - 8n - 192 &= 0 && \text{Reduce all terms by a factor of 4} \\
n^2 - 2n - 48 &= 0 && \text{Factor} \\
(n+6)(n-8) &= 0 && \text{Apply zero product rule} \\
n + 6 = 0 \text{ or } n - 8 &= 0 && \text{Isolate variable terms} \\
n = -6 \text{ or } n &= 8 && \text{Solutions}
\end{aligned}
$$

Since the quantity of students is always positive, we omit the solution $n = -6$ and there were 8 students in the original group.

9.5.1 Revenue Problems Homework

1. A merchant bought some pieces of silk for \$900. Had he bought 3 pieces more for the same money, he would have paid \$15 less for each piece. Find the number of pieces purchased.

2. A number of men subscribed a certain amount to make up a deficit of \$100 but 5 men failed to pay and thus increased the share of the others by \$1 each. Find the amount that each man paid.

3. A merchant bought a number of barrels of apples for \$120. He kept two barrels and sold the remainder at a profit of \$2 per barrel making a total profit of \$34. How many barrels did he originally buy?

4. A dealer bought a number of sheep for \$440. After 5 had died he sold the remainder at a profit of \$2 each making a profit of \$60 for the sheep. How many sheep did he originally purchase?

5. A man bought a number of articles at equal cost for \$500. He sold all but two for \$540 at a profit of \$5 for each item. How many articles did he buy?

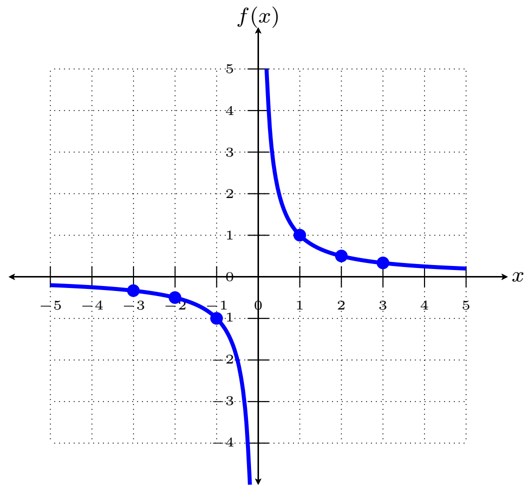
6. A clothier bought a lot of suits for \$750. He sold all but 3 of them for \$864 making a profit of \$7 on each suit sold. How many suits did he buy?

7. A group of boys bought a boat for \$450. Five boys failed to pay their share, hence each remaining boys were compelled to pay \$4.50 more. How many boys were in the original group and how much had each agreed to pay?

8. The total expenses of a camping party were \$72. If there had been 3 fewer persons in the party, it would have cost each person \$2 more than it did. How many people were in the party and how much did it cost each one?

9.6 Graphs of Rational Functions

Previously, in the chapters where we discussed functions, we had a function from the library $f(x) = \dfrac{1}{x}$. Recall, the graph of this function is



We plotted some points we obtained from the table and determined that the domain is all real numbers except for $x = 0$: $\{x | x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$. We called this function a ***rational function***.

Definition

A ***rational function***, $R(x)$, is a ratio of two polynomials, $P(x)$ and $Q(x)$, of the form

$$R(x) = \frac{P(x)}{Q(x)},$$

where $Q(x) \neq 0$.

Note

In this textbook, we only discuss when $P(x) = 1$ and when $Q(x)$ is of the form x^n, where n is a positive integer and

$$R(x) = \frac{1}{x^n}$$

For cases when $P(x)$ is a polynomial other than the constant function 1 and $Q(x)$ is a polynomial other than the *power function* x^n is left for future Algebra classes.

Let's investigate these functions a little further. We know the domain is all real numbers except for $x = 0$, but let's look at the graph more closely. Notice, in the graph of $f(x)$ above, the graph doesn't intersect the y axis. Why? Well, let's set $y = f(x) = 0$ and solve:

$$f(x) = 0 = \frac{1}{x}$$

When is this fraction zero? We know from previous sections that a fraction is zero when the numerator is zero. Will the numerator ever be zero, i.e.,

$$1 \stackrel{?}{=} 0$$

No, never! This means there are no values of x such that $y = 0$, and that $y = 0$ is not in the range of the function.

This is not a coincidence. The fact that $x \neq 0$ and $y \neq 0$ for the function $f(x) = \frac{1}{x}$ means that $f(x)$ has ***vertical*** and ***horizontal*** asymptotes at $x = 0$ and $y = 0$, respectively.

Definition

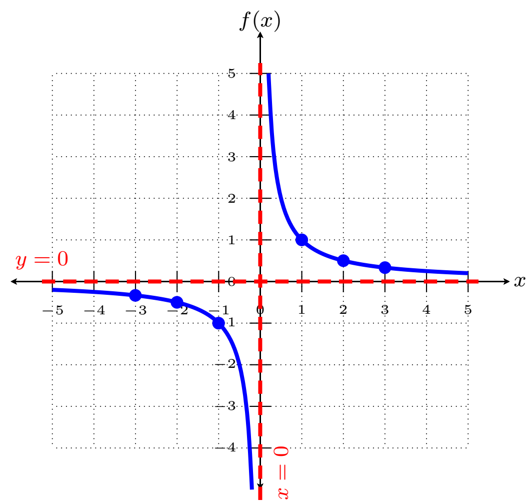
A function, $R(x)$, has a ***horizontal asymptote*** at $y = 0$ and a ***vertical asymptote*** at $x = 0$ when $R(x)$ is of the form

$$\frac{1}{x^n}$$

We denote these asymptotes by drawing *dashed* lines for lines $x = 0$ and $y = 0$.

Example 9.19

Let's regraph $f(x) = \frac{1}{x}$ showing the horizontal and vertical asymptotes at $y = 0$ and $x = 0$, respectively.



So, we see the asymptotes, in red, are dashed lines on the x and y-axis and are the lines $y = 0$ and $x = 0$. The only case in which the horizontal and vertical asymptotes move left or right, and up or down, respectively, is if there are shifts to the parent function $f(x)$.

Example 9.20

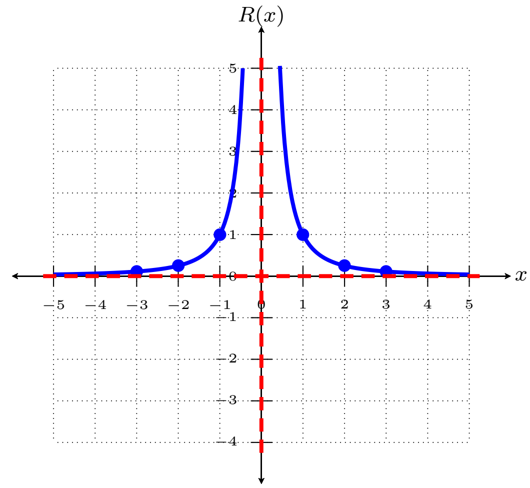
Graph $R(x) = \frac{1}{x^2}$.

Solution.

Let's pick x-coordinates, and find corresponding y-values.

x	$R(x) = \frac{1}{x^2}$	$(x, R(x))$
-3	$f(-3) = \frac{1}{(-3)^2}$	$(-3, \frac{1}{9})$
-2	$f(-2) = \frac{1}{(-2)^2}$	$(-2, \frac{1}{4})$
-1	$f(-1) = 1$	$(-1, 1)$
0	$f(0) =$ undefined	no point
1	$f(1) = 1$	$(1, 1)$
2	$f(2) = \frac{1}{2^2}$	$(2, \frac{1}{4})$
3	$f(3) = \frac{1}{3^2}$	$(3, \frac{1}{9})$

Plot the ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. The domain of $R(x)$ is all real numbers except for $x = 0$: $\{x|x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$. Since $R(x)$ has ***horizontal*** and ***vertical asymptotes*** at $y = 0$ and $x = 0$, respectively, let's draw the lines that represent these asymptotes.



9.6.1 Graphing Rational Functions Using Shifts

Let's take a look when rational functions' graphs contain horizontal and vertical shifts. It will be interesting to see the horizontal and vertical asymptotes change due to these shifts.

Horizontal and Verticals Shifts of Rational Functions

Given $R(x)$ is a rational function, a horizontal shift and vertical shift of $R(x)$ are described below:

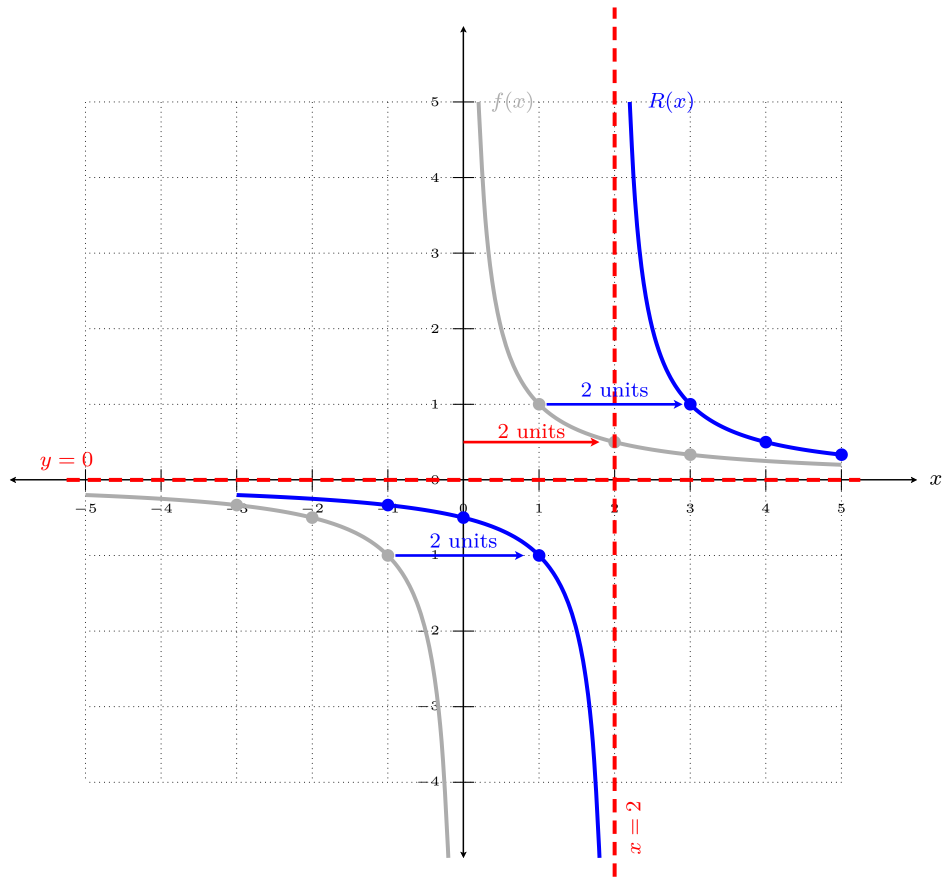
	$R(x-h)$	$R(x+h)$	$R(x)-k$	$R(x)+k$
Shift	Horizontal shift	Horizontal shift	Vertical shift	Vertical shift
Units	Shift h units to the right	Shift h units to the left	Shift k units downward	Shift k units upward
Asymptotes	Vertical asymptote is $x = h$ Horizontal asymptote is $y = 0$	Vertical asymptote is $x = h$ Horizontal asymptote is $y = 0$	Vertical asymptote is $x = 0$ Horizontal asymptote is $y = k$	Vertical asymptote is $x = 0$ Horizontal asymptote is $y = k$

Example 9.21

Graph $R(x) = \dfrac{1}{x-2}$.

Solution.

Let's start by taking the *parent function* $f(x) = \dfrac{1}{x}$. We see that $R(x) = f(x-2)$ because we replaced x with the factor $(x-2)$. Looking at the table above, we see this is a horizontal shift with $h = 2$, moving 2 units to the right, and the vertical asymptote changes to $x = 2$.



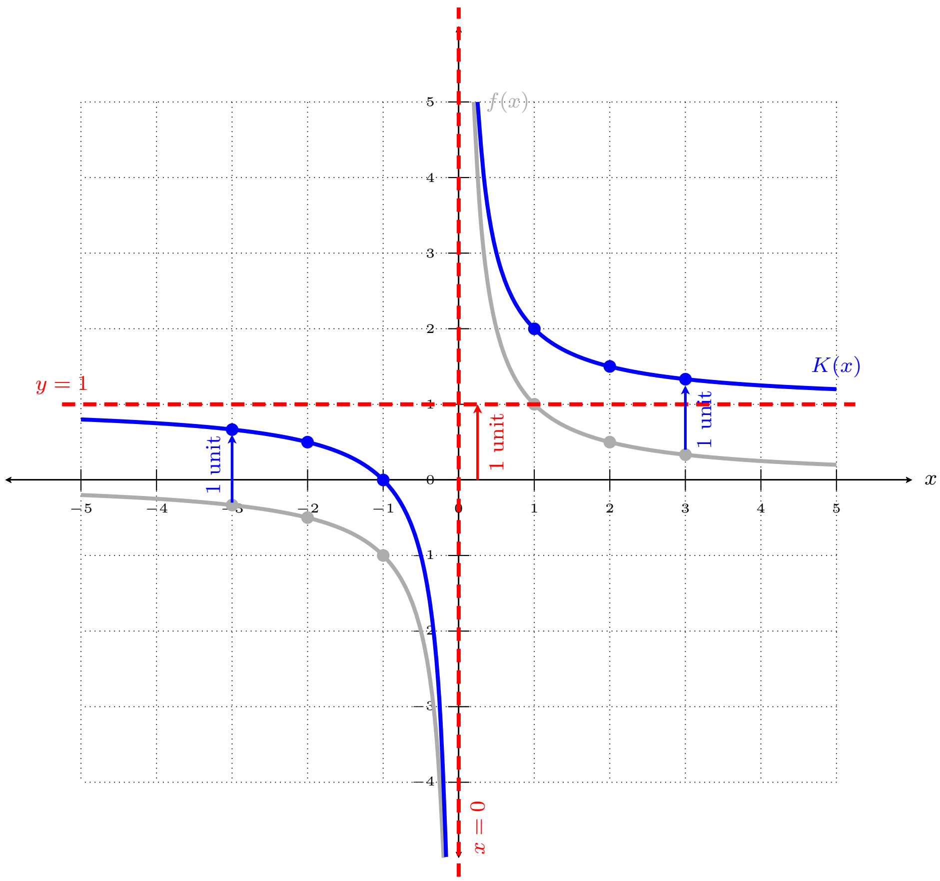
We can see the gray graph, $f(x)$, moved two units to the right, in addition to the vertical asymptote. Recall, from the table above, the horizontal asymptote stays $y = 0$. Hence, the blue graph, $R(x)$, is the final graph after applying the shifts.

Example 9.22

Graph $K(x) = \dfrac{1}{x} + 1$.

Solution.

Let's start by taking the *parent function* $f(x) = \dfrac{1}{x}$. We see that $K(x) = f(x) + 1$ because we added 1 to $f(x)$. Looking at the table above, we see this is a vertical shift with $k = 2$, moving 1 unit upward, and the horizontal asymptote changes to $y = 1$.



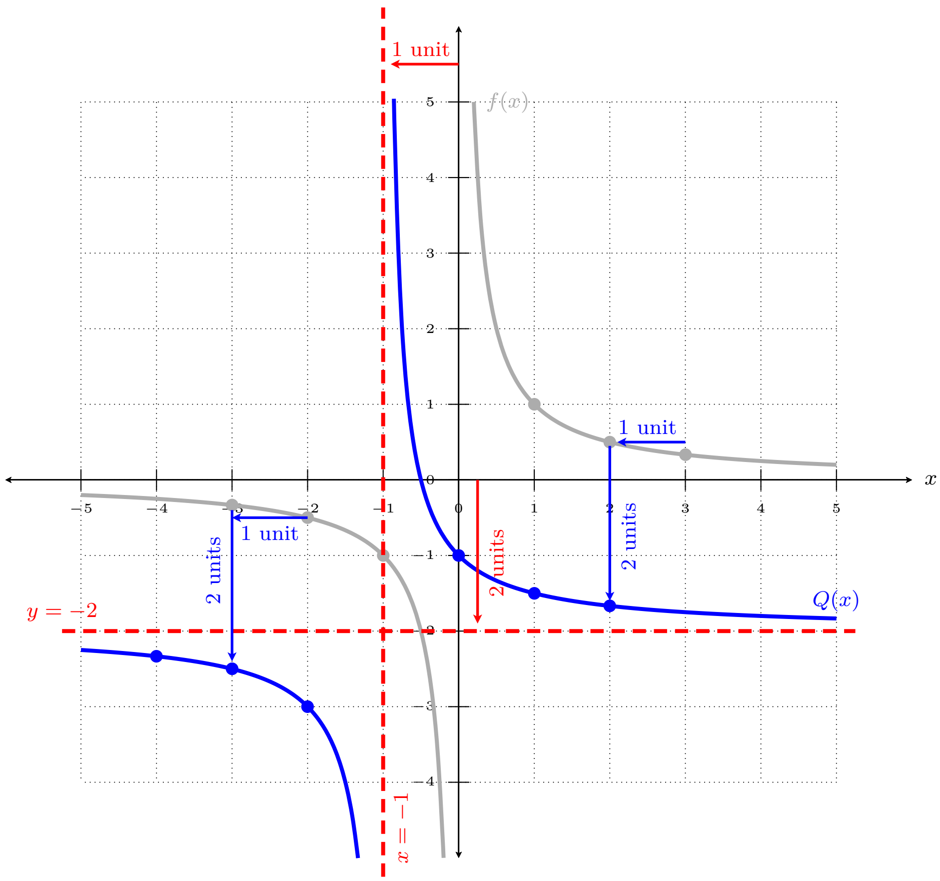
We can see the gray graph, $f(x)$, moved one unit upward, in addition to the horizontal asymptote. Recall, from the table above, the vertical asymptote stays $x = 0$. Hence, the blue graph, $K(x)$, is the final graph after applying the shifts.

Example 9.23

Graph $Q(x) = \dfrac{1}{x+1} - 2$.

Solution.

Let's start by taking the *parent function* $f(x) = \dfrac{1}{x}$. We see that $Q(x) = f(x+1) - 2$ because we replaced x with the factor $(x+1)$ and we subtracted 2 from $f(x)$. Looking at the table above, we see $Q(x)$ has a few shifts: a horizontal shift with $h = -1$, moving 1 unit to the left, a vertical shift with $k = -2$, moving 2 units downward, and the vertical and horizontal asymptotes change to $x = -1$ and $y = -2$, respectively.



We can see the gray graph, $f(x)$, moved one unit to the left and 2 units downward in addition to the horizontal and vertical asymptotes. Notice, we had a vertical and horizontal shift. We moved $f(x)$ one unit left, then 2 units down for all points. These shifts cause the asymptotes to move too. In fact, the vertical asymptote moved one unit to the left and the horizontal asymptote moved 2 units downward. Hence, the blue graph, $Q(x)$, is the final graph after applying the shifts.

9.6.2 Graphs Rational Functions Homework

Graph each rational functions using the parent function $f(x) = \frac{1}{x}$. Include the vertical and horizontal asymptotes.

1. $R(x) = \dfrac{1}{x-1}$

2. $Q(x) = \dfrac{1}{x+3}$

3. $S(x) = \dfrac{1}{x} + 2$

4. $T(x) = \dfrac{1}{x} - 4$

5. $U(x) = \dfrac{1}{x+2} - 1$

6. $U(x) = \dfrac{1}{x-3} - 2$

7. $P(x) = \dfrac{1}{x} + 3$

8. $N(x) = \dfrac{1}{x-1} - 4$

9.7 Rational Equations and Applications: Answers to the Homework Exercises

Rational Equations

1. $-\frac{1}{2}, \frac{2}{3}$
3. 2
5. -5
7. $\frac{16}{3}, 5$
9. $-\frac{1}{5}, 5$
11. $0, 5$
13. $\frac{2}{3}$
15. $-\frac{2}{3}$
17. -10
19. $-3, 1$
21. $\frac{1}{3}$
23. $5, 10$
25. 2
27. 10
29. -1
31. 1
33. 1

Rational Inequalities

1. $(-1, 3)$
3. $(-1, \infty)$
5. $(-\infty, 0) \cup (5, 9)$
7. $[-12, 1) \cup [4, \infty)$
9. $(-\infty, 0] \cup [2, 3)$
11. $(-\infty, -6] \cup (4, \infty)$

Work-rate Problems

1. 4 and 6 hours
3. 2 and 3 hours
5. 4 and 12 hours
7. $1\frac{1}{3}$ days
9. 8 days
11. 2 days
13. 9 hours
15. 16 hours
17. 15 hours
19. $5\frac{1}{4}$ minutes
21. 24 min
23. 6 and 12 hours
25. 7 and $17\frac{1}{2}$ minutes
27. 21 and 15 minutes

Uniform Motion Problems

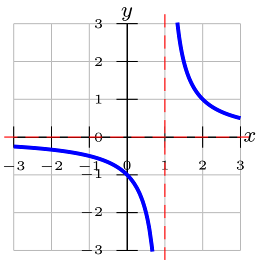
1. 60 km/hr, 80 km/hr
3. 200 km/hr
5. 3.033 km/hr
7. 30 mph, 40 mph
9. 36 mph
11. 40 mph, 60 mph
13. 4 mph

Revenue Problems

1. 12
3. 24
5. 20
7. 25 at $18 each

Graphs of Rational Functions

1.



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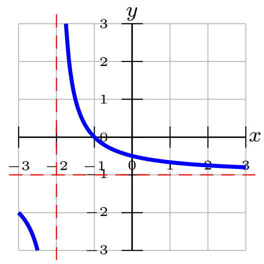

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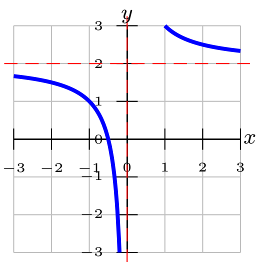

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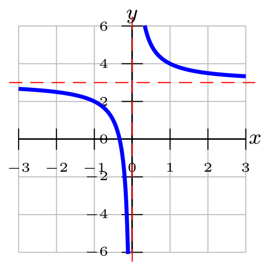


Chapter 10

Radicals

Chapter Objectives

By the end of this chapter, the student should be able to

- Simplify radical expressions
- Rationalize denominators (monomial and binomial) of radical expressions
- Add, subtract, and multiply radical expressions with and without variables
- Solve equations containing radicals and radical functions
- Solve equations containing rational exponents

Radicals are a common concept in algebra. In fact, we think of radicals as reversing the operation of an exponent. Hence, instead of the "square" of a number, we take the "square root" a number; instead of the "cube" of a number, we take the "cube root" a number, and so on. Square roots are the most common type of radical used in algebra.

World Note

The radical sign, when first used, was an R with a line through the tail, ℞, similar to our medical prescription symbol. The R came from the latin, "radix", which can be translated as "source" or "foundation." It wasn't until the 1500s that our current symbol was first used in Germany, but even then it was just a check mark with no bar over the numbers, √.

Definition

If a is a positive real number, then the ***principal square root*** of a number a is defined as

$$\sqrt{a} = b \text{ if and only if } a = b^2,$$

where $b > 0$. The $\sqrt{}$ is the radical symbol, and a is called the ***radicand***.

If given something like $\sqrt[3]{a}$, then 3 is called the ***root*** or ***index***; hence, $\sqrt[3]{a}$ is called the *cube root* or *third root* of a. In general,

$$\sqrt[n]{a} = b \text{ if and only if } a = b^n$$

If n is even, then a and b must be greater than or equal to zero. If n is odd, then a and b can be any real number.

Example 10.1

Here are some examples of square roots:

$$\sqrt{1} = 1 \qquad \sqrt{121} = 11$$

$$\sqrt{4} = 2 \qquad \sqrt{625} = 25$$

$$\sqrt{9} = 3 \qquad \sqrt{-81} = \text{not a real number}$$

The final example, $\sqrt{-81}$ is *not a real number*. There is a future section in which will discuss examples like $\sqrt{-81}$. Recall, if the root is even, then the radicand must be greater than or equal to zero and since $-81 < 0$, then there is no real number in which we can square and will result in -81, i.e., $?^2 = -81$. So, for now, when we obtain a radicand that is negative and the root is even, we say that this number is not a real number. There is a type of number where we can evaluate these numbers, but just not a real one.

10.1 Simplify radicals

Not all radicands are perfect squares, where when we take the square root, we obtain a positive integer. For example, if we input $\sqrt{8}$ in a calculator, the calculator would display 2.8284271247461900976033774484193...and even this number is a rounded approximation of the square root. To be as accurate as possible, we will leave all answers in exact form, i.e., answers contain integers and radicals- no decimals.

✎ **Note**

When we say to ***simplify an expression*** with radicals, the simplified expression should have

- a radical, unless the radical reduces to an integer
- a radicand with no factors containing perfect squares
- no decimals

Following these guidelines ensures the expression is in its simplest form.

10.1.1 Simplify radicals

Product rule for radicals

If a, b are any two positive real numbers, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

In general, if a, b are any two positive real numbers, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b},$$

where n is a positive integer and $n \geq 2$.

Example 10.2

Simplify: $\sqrt{75}$

Solution.

We can apply the product rule for radicals to simplify this number. We need to find the largest factor of 75 that is a perfect square (since we have a square root) and rewrite the radicand as a product of this perfect square and its other factor. The largest factor of radicand 75 that is a perfect square is 25.

$$\begin{aligned} \sqrt{75} &\quad \text{Rewrite radicand as a product of 25 and 3} \\ \sqrt{25 \cdot 3} &\quad \text{Apply product rule for radicals} \\ \sqrt{25} \cdot \sqrt{3} &\quad \text{Simplify each square root} \\ 5 \cdot \sqrt{3} &\quad \text{Rewrite} \\ 5\sqrt{3} &\quad \text{Simplified expression} \end{aligned}$$

If the radicand is not a perfect square, we leave as is; hence, we left $\sqrt{3}$ as is.

Example 10.3

Simplify: $\sqrt{72}$

Solution.

We can apply the product rule for radicals to simplify this number. We need to find the largest factor of 72 that is a perfect square (since we have a square root) and rewrite the radicand as a product of this perfect square and its other factor. The largest factor of radicand 72 that is a perfect square is 36.

$$\begin{aligned} \sqrt{72} &\quad \text{Rewrite radicand as a product of 36 and 2} \\ \sqrt{36 \cdot 2} &\quad \text{Apply product rule for radicals} \\ \sqrt{36} \cdot \sqrt{2} &\quad \text{Simplify each square root} \\ 6 \cdot \sqrt{2} &\quad \text{Rewrite} \\ 6\sqrt{2} &\quad \text{Simplified expression} \end{aligned}$$

If the radicand is not a perfect square, we leave as is; hence, we left $\sqrt{2}$ as is.

10.1.2 Simplify radicals with coefficients

Example 10.4

Simplify: $5\sqrt{63}$

Solution.

We can apply the product rule for radicals to simplify this number and multiply coefficients in the last steps. We need to find the largest factor of 63 that is a perfect square (since we have a square root) and rewrite the radicand as a product of this perfect square and its other factor. The largest factor of radicand 63 that is a perfect square is 9.

$$\begin{aligned} 5\sqrt{63} &\quad \text{Rewrite radicand as a product of 9 and 7} \\ 5\sqrt{9 \cdot 7} &\quad \text{Apply product rule for radicals} \\ 5 \cdot \sqrt{9} \cdot \sqrt{7} &\quad \text{Simplify each square root} \\ 5 \cdot 3 \cdot \sqrt{7} &\quad \text{Rewrite and simplify coefficients} \\ 15\sqrt{7} &\quad \text{Simplified expression} \end{aligned}$$

> If the radicand is not a perfect square, we leave as is; hence, we left $\sqrt{7}$ as is.

10.1.3 Rational exponents

When we simplify radicals, we extract roots of factors with exponents in which are multiples of the root (index). For example, $\sqrt{x^4} = \sqrt[2]{x^4} = x^2$, but notice we just divided the power on x by the root. Let's look at the example again, but now as division of exponents:

$$\sqrt{x^4} = \sqrt[2]{x^4} = x^{\frac{4}{2}} = x^2$$

Division with exponents, or *fraction exponents*, are called ***rational exponents***.

Definition

Let a be the base, and m and n be real real numbers. Then

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

The denominator of a ***rational exponent*** is the root on the radical and vice versa.

Example 10.5

Rewrite each radical with its corresponding rational exponent.

a) $\left(\sqrt[5]{x}\right)^3$ **b)** $\left(\sqrt[6]{3x}\right)^5$ **c)** $\dfrac{1}{\left(\sqrt[7]{a}\right)^3}$ **d)** $\dfrac{1}{\left(\sqrt[3]{xy}\right)^2}$

Solution.

a) For the expression $\left(\sqrt[5]{x}\right)^3$, we see the root is 5. This means that the denominator of the rational exponent is 5. Hence, the numerator is the exponent 3: $\left(\sqrt[5]{x}\right)^3 = x^{\frac{3}{5}}$.

b) For the expression $\left(\sqrt[6]{3x}\right)^5$, we see the root is 6. This means that the denominator of the rational exponent is 6. Hence, the numerator is the exponent 5: $\left(\sqrt[6]{3x}\right)^5 = (3x)^{\frac{5}{6}}$.

c) For the expression $\dfrac{1}{\left(\sqrt[7]{a}\right)^3}$, we see the root is 7. This means that the denominator of the rational exponent is 7. Hence, the numerator is the exponent 3. Furthermore, since the expression with the radical is in the denominator, we can rewrite the expression using a negative exponent: $\dfrac{1}{\left(\sqrt[7]{a}\right)^3} = (a)^{-\frac{3}{7}}$.

d) For the expression $\dfrac{1}{\left(\sqrt[3]{xy}\right)^2}$, we see the root is 3. This means that the denominator of the rational exponent is 3. Hence, the numerator is the exponent 2. Furthermore, since the expression with the radical is in the denominator, we can rewrite the expression using a negative exponent: $\dfrac{1}{\left(\sqrt[3]{xy}\right)^2} = (xy)^{-\frac{2}{3}}$.

Example 10.6

Rewrite each expression in its equivalent radical form.

a) $a^{\frac{5}{3}}$ **b)** $(2mn)^{\frac{2}{7}}$ **c)** $x^{-\frac{4}{5}}$ **d)** $(xy)^{-\frac{2}{9}}$

Solution.

a) From the definition, we know that the denominator of the rational exponent is the root making the numerator the power: $a^{\frac{5}{3}} = \sqrt[3]{a^5}$ or $\left(\sqrt[3]{a}\right)^5$.

b) From the definition, we know that the denominator of the rational exponent is the root making the numerator the power: $(2mn)^{\frac{2}{7}} = \sqrt[7]{(2mn)^2}$ or $\left(\sqrt[7]{2mn}\right)^2$.

c) From the definition, we know that the denominator of the rational exponent is the root making the numerator the power: $x^{-\frac{4}{5}} = \left(\sqrt[5]{x}\right)^{-4}$. Notice that the expression still contains a negative exponent. Hence, we need to reciprocate the radical to rewrite the expression with only positive exponents:

$$x^{-\frac{4}{5}} = \frac{1}{\left(\sqrt[5]{x}\right)^4}$$

d) From the definition, we know that the denominator of the rational exponent is the root making the numerator the power: $(xy)^{-\frac{2}{9}} = \left(\sqrt[9]{x}\right)^{-2}$. Notice that the expression still contains a negative exponent. Hence, we need to reciprocate the radical to rewrite the expression with only positive exponents:

$$(xy)^{-\frac{2}{9}} = \frac{1}{\left(\sqrt[9]{xy}\right)^2}$$

World Note

Nicole Oresme, a Mathematician born in Normandy was the first to use rational exponents. He used the notation $\frac{1}{3} \bullet 9^p$ to represent $9^{\frac{1}{3}}$. However, his notation went largely unnoticed.

The ability to change between rational exponential expressions and radical expressions allows us to evaluate expressions.

Example 10.7

Evaluate $27^{-\frac{4}{3}}$.

Solution.

We first rewrite the expression with only positive exponents, then evaluate the exponent.

$27^{-\frac{4}{3}}$	Rewrite the expression with positive exponents
$\frac{1}{27^{\frac{4}{3}}}$	Rewrite in radical form
$\frac{1}{\left(\sqrt[3]{27}\right)^4}$	Evaluate radical $\sqrt[3]{27} = 3$
$\frac{1}{(3)^4}$	Evaluate exponent $3^4 = 81$
$\frac{1}{81}$	Result

Thus, $27^{-\frac{4}{3}} = \dfrac{1}{81}$. This result should emphasize the fact that negative exponents means reciprocals, and not negative numbers.

10.1.4 Simplify radicals with variables

Commonly, radicands can contain variables. When taking the square roots of variables, we know the root is 2; we do not always write it, but we know it's there. Hence, we apply the product rule of radicals by rewriting the variable's exponent and rewrite the exponents so that one of the exponents is the largest even number.

Example 10.8

Simplify: $\sqrt{x^6y^5}$

Solution.

We can apply the product rule for radicals to simplify by rewriting the variable's exponent and rewrite the exponents so that one of the exponents is the largest even number.

$$\begin{array}{rl} \sqrt{x^6y^5} & \text{Rewrite radicand} \\ \sqrt{x^6 \cdot y^4 \cdot y^1} & \text{Apply product rule for radicals} \\ \sqrt{x^6} \cdot \sqrt{y^4} \cdot \sqrt{y} & \text{Simplify each square root} \\ x^3 \cdot y^2 \cdot \sqrt{y} & \text{Rewrite and simplify coefficients} \\ x^3y^2\sqrt{y} & \text{Simplified expression} \end{array}$$

Notice that $\left(x^3\right)^2 = x^6$ and $\left(y^2\right)^2 = y^4$; hence, we extract the perfect squares of the variables and leave the $\sqrt{y}$ as is.

Note

Recall, when taking a square root of a number, the radicand must be greater than or equal to zero. So, when we are applying the square root to variables, the variables must also be greater than or equal to zero.

Notice, we are essentially dividing the exponents on the variables by two and the factor that remains in the radicand has exponent 1.

Example 10.9

Simplify: $-5\sqrt{18x^4y^6z^{10}}$. *Assume all variables are positive.*

Solution.

We can apply the product rule for radicals to simplify by rewriting the variable's exponent and rewrite the exponents so that one of the exponents is the largest even number.

$-5\sqrt{18x^4y^6z^{10}}$	Rewrite radicand
$-5\cdot\sqrt{9\cdot 2\cdot x^4\cdot y^6\cdot z^{10}}$	Apply product rule for radicals
$-5\cdot\sqrt{9}\cdot\sqrt{2}\cdot\sqrt{x^4}\cdot\sqrt{y^6}\cdot\sqrt{z^{10}}$	Simplify each square root
$-5\cdot 3\cdot\sqrt{2}\cdot x^2\cdot y^3\cdot z^5$	Rewrite and simplify coefficients
$-15x^2y^3z^5\sqrt{2}$	Simplified expression

Example 10.10

Simplify: $\sqrt{20x^5y^9z^6}$. *Assume all variables are positive.*

Solution.

We can apply the product rule for radicals to simplify by rewriting the variable's exponent and rewrite the exponents so that one of the exponents is the largest even number.

$\sqrt{20x^5y^9z^6}$	Rewrite radicand
$\sqrt{4\cdot 5\cdot x^4\cdot x\cdot y^8\cdot y\cdot z^6}$	Apply product rule for radicals
$\sqrt{4}\cdot\sqrt{5}\cdot\sqrt{x^4}\cdot\sqrt{x}\cdot\sqrt{y^8}\cdot\sqrt{y}\cdot\sqrt{z^6}$	Simplify each square root
$2\cdot\sqrt{5}\cdot x^2\cdot\sqrt{x}\cdot y^4\cdot\sqrt{y}\cdot z^3$	Rewrite and simplify coefficients
$2x^2y^4z^3\sqrt{5xy}$	Simplified expression

10.1.5 Simplify Radicals Homework

Simplify. *Assume all variables are positive.*

1. $\sqrt{245}$ **2.** $\sqrt{36}$ **3.** $\sqrt{12}$

4. $3\sqrt{12}$ **5.** $6\sqrt{128}$ **6.** $-8\sqrt{392}$

7. $\sqrt{192n}$ **8.** $\sqrt{196v^2}$ **9.** $\sqrt{252x^2}$

10. $-\sqrt{100k^4}$ **11.** $-7\sqrt{64x^4}$ **12.** $-5\sqrt{36m}$

13. $\sqrt{45x^2y^2}$ **14.** $\sqrt{16x^3y^3}$ **15.** $\sqrt{320x^4y^4}$

16. $6\sqrt{80xy^2}$ **17.** $5\sqrt{245x^2y^3}$ **18.** $-2\sqrt{180u^3v}$

19. $-8\sqrt{180x^4y^2z^4}$ **20.** $2\sqrt{80hj^4k}$ **21.** $-4\sqrt{54mnp^2}$

22. $\sqrt{125}$ **23.** $\sqrt{196}$ **24.** $\sqrt{338}$

25. $5\sqrt{32}$ **26.** $7\sqrt{128}$ **27.** $-7\sqrt{63}$

28. $\sqrt{343b}$ **29.** $\sqrt{100n^3}$ **30.** $\sqrt{200a^3}$

31. $-4\sqrt{175p^4}$ **32.** $-2\sqrt{128n}$ **33.** $8\sqrt{112p^2}$

34. $\sqrt{72a^3b^4}$ **35.** $\sqrt{512a^4b^2}$ **36.** $\sqrt{512m^4n^3}$

37. $8\sqrt{98mn}$ **38.** $2\sqrt{72x^2y^2}$ **39.** $-5\sqrt{72x^3y^4}$

40. $6\sqrt{50a^4bc^2}$ **41.** $-\sqrt{32xy^2z^3}$ **42.** $-8\sqrt{32m^2p^4q}$

Write each expression in radical form with only positive exponents.

43. $m^{\frac{3}{5}}$ **44.** $(7x)^{\frac{3}{2}}$ **45.** $(10r)^{-\frac{3}{4}}$ **46.** $(6b)^{-\frac{4}{3}}$

Write each expression in exponential form.

47. $\dfrac{1}{(\sqrt{6x})^3}$ **48.** $\dfrac{1}{(\sqrt[4]{n})^7}$ **49.** $\sqrt{v}$ **50.** $\sqrt{5a}$

Evaluate without using a calculator.

51. $8^{\frac{2}{3}}$ **52.** $4^{\frac{3}{2}}$ **53.** $16^{\frac{1}{4}}$ **54.** $100^{-\frac{3}{2}}$

10.2 Add and subtract radicals

Adding and subtracting radicals are very similar to adding and subtracting with variables. In order to combine terms, they need to be like terms. With radicals, we have something similar called *like radicals*. Let's look at an example with like terms and like radicals.

$$\begin{array}{cc} 2x+5x & 2\sqrt{3}+5\sqrt{3} \\ (2+5)x & (2+5)\sqrt{3} \\ 7x & 7\sqrt{3} \end{array}$$

Notice that when we combined the terms with $\sqrt{3}$, it was similar to combining terms with x. When adding and subtracting with radicals, we can combine *like radicals* just as like terms.

Definition

If two radicals have the same radicand and the same root, then they are called ***like radicals***. If this is so, then

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x},$$

where a, b are real numbers and x is some positive real number.

In general, for any root n,

$$a\sqrt[n]{x} \pm b\sqrt[n]{x} = (a \pm b)\sqrt[n]{x},$$

where a, b are real numbers and x is some positive real number.

Note

When simplifying radicals with addition and subtraction, we will simplify the expression first, then extract out any factors from the radicand following the guidelines in the previous section.

10.2.1 Add and subtract like radicals

Example 10.11

Simplify: $7\sqrt[5]{6}+4\sqrt[5]{3}-9\sqrt[5]{3}+\sqrt[5]{6}$

Solution.

Notice, all the indices are the same, but two of the radicands are different. We only combine *like radicals*, where the root and radicand are the same.

$$\begin{array}{rl} 7\sqrt[5]{6}+4\sqrt[5]{3}-9\sqrt[5]{3}+\sqrt[5]{6} & \text{Combine the like radicals} \\ (7+1)\sqrt[5]{6}+(4-9)\sqrt[5]{3} & \text{Simplify} \\ 8\sqrt[5]{6}-5\sqrt[5]{3} & \text{Simplified expression} \end{array}$$

Notice, radicands 6 and 3 have no factors that are perfect 5^{th} powers. Thus, the expression is completely simplified.

10.2.2 Simplify, then add and subtract like radicals

Example 10.12

Simplify: $5\sqrt{45}+6\sqrt{18}-2\sqrt{98}+\sqrt{20}$

Solution.

Notice, all the indices are the same, but none of the radicands are the same. However, we can see that the radicands have factors that are perfect squares. We can simplify the radicands first, then see if we can combine *like radicals*.

$$\begin{aligned}
5\sqrt{45}+6\sqrt{18}-2\sqrt{98}+\sqrt{20} &\quad \text{Rewrite radicand}\\
5\cdot\sqrt{9\cdot 5}+6\cdot\sqrt{9\cdot 2}-2\cdot\sqrt{49\cdot 2}+\sqrt{4\cdot 5} &\quad \text{Apply product rule for radicals}\\
5\cdot\sqrt{9}\cdot\sqrt{5}+6\cdot\sqrt{9}\cdot\sqrt{2}-2\cdot\sqrt{49}\cdot\sqrt{2}+\sqrt{4}\cdot\sqrt{5} &\quad \text{Simplify each square root}\\
5\cdot 3\cdot\sqrt{5}+6\cdot 3\cdot\sqrt{2}-2\cdot 7\cdot\sqrt{2}+2\cdot\sqrt{5} &\quad \text{Rewrite and simplify coefficients}\\
15\sqrt{5}+18\sqrt{2}-14\sqrt{2}+2\sqrt{5} &\quad \text{Combine the like radicals}\\
(15+2)\sqrt{5}+(18-14)\sqrt{2} &\quad \text{Simplify}\\
17\sqrt{5}+4\sqrt{2} &\quad \text{Simplified expression}
\end{aligned}$$

World Note

The Arab writers of the 16th century used the symbol similar to the greater than symbol with a dot underneath, $\underset{\cdot}{<}$, for radicals.

Example 10.13

Simplify: $4\sqrt[3]{54}-9\sqrt[3]{16}+5\sqrt[3]{9}$

Solution.

We apply the same method as the previous examples, but the root is 3 and we will look for the largest factor of the radicand that is a perfect cube when simplifying the radicals.

$$\begin{aligned}
4\sqrt[3]{54}-9\sqrt[3]{16}+5\sqrt[3]{9} &\quad \text{Rewrite radicand}\\
4\cdot\sqrt[3]{27\cdot 2}-9\cdot\sqrt[3]{8\cdot 2}+5\cdot\sqrt[3]{9} &\quad \text{Apply product rule for radicals and simplify}\\
4\cdot 3\sqrt[3]{2}-9\cdot 2\sqrt[3]{2}+5\sqrt[3]{9} &\quad \text{Rewrite and simplify coefficients}\\
12\sqrt[3]{2}-18\sqrt[3]{2}+5\sqrt[3]{9} &\quad \text{Combine the like radicals}\\
(12-18)\sqrt[3]{2}+5\sqrt[3]{9} &\quad \text{Simplify}\\
-6\sqrt[3]{2}+5\sqrt[3]{9} &\quad \text{Simplified expression}
\end{aligned}$$

10.2.3 Add and Subtract Radicals Homework

Simplify.

1. $2\sqrt{5}+2\sqrt{5}+2\sqrt{5}$

2. $-3\sqrt{2}+3\sqrt{5}+3\sqrt{5}$

3. $-2\sqrt{6}-2\sqrt{6}-\sqrt{6}$

4. $3\sqrt{6}+3\sqrt{5}+2\sqrt{5}$

5. $2\sqrt{2}-3\sqrt{18}-\sqrt{2}$

6. $-3\sqrt{6}-\sqrt{12}+3\sqrt{3}$

7. $3\sqrt{2}+2\sqrt{8}-3\sqrt{18}$

8. $3\sqrt{18}-\sqrt{2}-3\sqrt{2}$

9. $-3\sqrt{6}-3\sqrt{6}-\sqrt{3}+3\sqrt{6}$

10. $-2\sqrt{18}-3\sqrt{8}-\sqrt{20}+2\sqrt{20}$

11. $-2\sqrt{24}-2\sqrt{6}+2\sqrt{6}+2\sqrt{20}$

12. $3\sqrt{24}-3\sqrt{27}+2\sqrt{6}+2\sqrt{8}$

13. $-2\sqrt[3]{16}+2\sqrt[3]{16}+2\sqrt[3]{2}$

14. $2\sqrt[4]{243}-2\sqrt[4]{243}-\sqrt[4]{3}$

15. $3\sqrt[4]{2}-2\sqrt[4]{2}-\sqrt[4]{243}$

16. $-\sqrt[4]{324}+3\sqrt[4]{324}-3\sqrt[4]{4}$

17. $2\sqrt[4]{2}+2\sqrt[4]{3}+3\sqrt[4]{64}-\sqrt[4]{3}$

18. $-3\sqrt[5]{6}-\sqrt[5]{64}+2\sqrt[5]{192}-2\sqrt[5]{64}$

19. $2\sqrt[5]{160}-2\sqrt[5]{192}-\sqrt[5]{160}-\sqrt[5]{-160}$

20. $-\sqrt[6]{256}-2\sqrt[6]{4}-3\sqrt[6]{320}-2\sqrt[6]{128}$

21. $-3\sqrt{6}-3\sqrt{3}-2\sqrt{3}$

22. $-2\sqrt{6}-\sqrt{3}-3\sqrt{6}$

23. $-3\sqrt{3}+2\sqrt{3}-2\sqrt{3}$

24. $-\sqrt{5}+2\sqrt{3}-2\sqrt{3}$

25. $-\sqrt{54}-3\sqrt{6}+3\sqrt{27}$

26. $-\sqrt{5}-\sqrt{5}-2\sqrt{54}$

27. $2\sqrt{20}+2\sqrt{20}-\sqrt{3}$

28. $-3\sqrt{27}+2\sqrt{3}-\sqrt{12}$

29. $-2\sqrt{2}-\sqrt{2}+3\sqrt{8}+3\sqrt{6}$

30. $-3\sqrt{18}-\sqrt{8}+2\sqrt{8}+2\sqrt{8}$

31. $-3\sqrt{8}-\sqrt{5}-3\sqrt{6}+2\sqrt{18}$

32. $2\sqrt{6}-\sqrt{54}-3\sqrt{27}-\sqrt{3}$

33. $3\sqrt[3]{135}-\sqrt[3]{81}-\sqrt[3]{135}$

34. $-3\sqrt[4]{4}+3\sqrt[4]{324}+2\sqrt[4]{64}$

35. $2\sqrt[4]{6}+2\sqrt[4]{4}+3\sqrt[4]{6}$

36. $-2\sqrt[4]{243}-\sqrt[4]{96}+2\sqrt[4]{96}$

37. $2\sqrt[4]{48}-3\sqrt[4]{405}-3\sqrt[4]{48}-\sqrt[4]{162}$

38. $-3\sqrt[7]{3}-3\sqrt[7]{768}+2\sqrt[7]{384}+3\sqrt[7]{5}$

39. $-2\sqrt[7]{256}-2\sqrt[7]{256}-3\sqrt[7]{2}-\sqrt[7]{640}$

10.3 Multiply and divide radicals

When we multiply radicals, we recall the product rule for radicals. As long as the roots of each radical in the product are the same, we can apply the product rule and then simplify as usual. At first, we will bring the radicals together under one radical, then simplify the radical by applying the product rule again.

Product rule for radicals

If a, b are any two positive real numbers, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

In general, if a, b are any two positive real numbers, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b},$$

where n is a positive integer and $n \geq 2$.

Example 10.14

Simplify: $-5\sqrt{14} \cdot 4\sqrt{6}$

Solution.

Notice both radicals are square roots and so, we can apply the product rule. Let's rewrite the product so that coefficients are with coefficients and radicals are with radicals:

$$\begin{array}{rl}
-5\sqrt{14} \cdot 4\sqrt{6} & \text{Rewrite} \\
-5 \cdot 4 \cdot \sqrt{14} \cdot \sqrt{6} & \text{Apply the product rule} \\
-5 \cdot 4 \cdot \sqrt{14 \cdot 6} & \text{Multiply} \\
-20\sqrt{84} & \text{Simplify } \sqrt{84} \\
-20\sqrt{4 \cdot 21} & \text{Apply the product rule} \\
-20 \cdot 2\sqrt{21} & \text{Multiply coefficients} \\
-40\sqrt{21} & \text{Product}
\end{array}$$

Example 10.15

Simplify: $2\sqrt[3]{18} \cdot 6\sqrt[3]{15}$

Solution.

Notice both radicals are cube roots and so, we can apply the product rule. Let's rewrite the product so that coefficients are with coefficients and radicals are with radicals:

$2\sqrt[3]{18}\cdot 6\sqrt[3]{15}$	Rewrite
$2\cdot 6\cdot \sqrt[3]{18}\cdot \sqrt[3]{15}$	Apply the product rule
$2\cdot 6\cdot \sqrt[3]{18\cdot 15}$	Multiply
$12\sqrt[3]{270}$	Simplify $\sqrt[3]{270}$
$12\sqrt[3]{27\cdot 10}$	Apply the product rule
$12\cdot 3\sqrt[3]{10}$	Multiply coefficients
$36\sqrt[3]{10}$	Product

10.3.1 Multiply radicals with monomials

Here we begin to multiply radicals with variables. In this section, we assume all variables to be positive.

Example 10.16

Simplify: $\sqrt[5]{8x^2}\cdot\sqrt[5]{4x^3}$

Solution.

Notice both radicals are fifth roots and so, we can apply the product rule.

$\sqrt[5]{8x^2}\cdot\sqrt[5]{4x^3}$	Apply the product rule
$\sqrt[5]{8x^2\cdot 4x^3}$	Multiply
$\sqrt[5]{32x^5}$	Simplify
$\sqrt[5]{2^5\cdot x^5}$	Apply the product rule
$2x$	Product

Example 10.17

Simplify: $\sqrt{60x^4}\cdot\sqrt{6x^7}$

Solution.

Notice both radicals are square roots and so, we can apply the product rule.

$\sqrt{60x^4}\cdot\sqrt{6x^7}$	Apply the product rule
$\sqrt{60x^4\cdot 6x^7}$	Multiply
$\sqrt{360x^{11}}$	Simplify
$\sqrt{36\cdot 10\cdot x^4\cdot x}$	Apply the product rule
$6\cdot x^2\cdot\sqrt{10\cdot x}$	Rewrite
$6x^2\sqrt{10x}$	Product

10.3.2 Distribute with radicals

When there is a factor in front of the parenthesis, we distribute that term to each inside the parenthesis. This method is applied to radicals. Recall, methods never change, just problems. Take the following example:

$2x(5y+3)$	$2\sqrt{7}\left(5\sqrt{3}+3\right)$
$2x\cdot 5y+2x\cdot 3$	$2\sqrt{7}\cdot 5\sqrt{3}+2\sqrt{7}\cdot 3$
$10xy+6x$	$10\sqrt{21}+6\sqrt{7}$

Notice, we distribute in the same sense as if we were in the polynomial chapter. Let's take a look at more examples. Recall, we assume all variables are positive.

Example 10.18

Simplify: $7\sqrt{6}\left(3\sqrt{10}-5\sqrt{15}\right)$

Solution.

$7\sqrt{6}\left(3\sqrt{10}-5\sqrt{15}\right)$	Distribute
$7\sqrt{6}\cdot 3\sqrt{10}-7\sqrt{6}\cdot 5\sqrt{15}$	Apply the product rule
$21\sqrt{60}-35\sqrt{90}$	Simplify each term as usual
$21\sqrt{4\cdot 15}-35\sqrt{9\cdot 10}$	Apply the product rule
$21\cdot 2\sqrt{15}-35\cdot 3\sqrt{10}$	Multiply coefficients
$42\sqrt{15}-105\sqrt{10}$	Simplified expression

Note, if the final expression had like radicals, then we would combine like radicals. Even though this resulted in unlike radicals, we continue to add or subtract radicals as usual.

Example 10.19

Simplify: $\sqrt{3}\left(7\sqrt{15x^3}+8x\sqrt{60x}\right)$

Solution.

$\sqrt{3}\left(7\sqrt{15x^3}+8x\sqrt{60x}\right)$	Distribute
$\sqrt{3}\cdot 7\sqrt{15x^3}+\sqrt{3}\cdot 8x\sqrt{60x}$	Apply the product rule
$7\sqrt{45x^3}+8x\sqrt{180x}$	Simplify each term as usual
$7\sqrt{9\cdot 5\cdot x^2\cdot x}+8x\sqrt{36\cdot 5\cdot x}$	Apply the product rule
$7\cdot 3x\sqrt{5x}+8x\cdot 6\sqrt{5x}$	Multiply coefficients
$21x\sqrt{5x}+48x\sqrt{5x}$	Combine like radicals
$69x\sqrt{5x}$	Simplified expression

10.3.3 Multiply radicals using FOIL

We can use the method of FOIL to multiply radicals that take the form of the "product of two binomials." Even though the factors aren't exactly two binomials, but the expression shares that form. Recall, we are only using the method of FOIL. Again, methods never change, just problems. We continue to assume all variables are positive.

Example 10.20

Simplify: $\left(\sqrt{5}-2\sqrt{3}\right)\left(4\sqrt{10}+6\sqrt{6}\right)$

Solution.

$$
\begin{array}{rl}
(\sqrt{5}-2\sqrt{3})(4\sqrt{10}+6\sqrt{6}) & \text{FOIL} \\
\underbrace{\sqrt{5}\cdot 4\sqrt{10}}_{\text{F}}+\underbrace{\sqrt{5}\cdot 6\sqrt{6}}_{\text{O}}-\underbrace{2\sqrt{3}\cdot 4\sqrt{10}}_{\text{I}}-\underbrace{2\sqrt{3}\cdot 6\sqrt{6}}_{\text{L}} & \text{Simplify and apply the product rule} \\
4\sqrt{50}+6\sqrt{30}-8\sqrt{30}-12\sqrt{18} & \text{Simplify each term as usual} \\
4\sqrt{25\cdot 2}+6\sqrt{30}-8\sqrt{30}-12\sqrt{9\cdot 2} & \text{Apply the product rule} \\
4\cdot 5\sqrt{2}+6\sqrt{30}-8\sqrt{30}-12\cdot 3\sqrt{2} & \text{Multiply coefficients} \\
20\sqrt{2}+6\sqrt{30}-8\sqrt{30}-36\sqrt{2} & \text{Combine like radicals} \\
-16\sqrt{2}-2\sqrt{30} & \text{Simplified expression}
\end{array}
$$

World Note

Clay tablets have been discovered revealing much about Babylonian mathematics dating back from 1800 to 1600 BC. In one of the tables, there is an approximation of $\sqrt{2}$ accurate to five decimal places: 1.41421.

10.3.4 Multiply radicals with special-product formulas

Example 10.21

Simplify: $(5\sqrt{7}+\sqrt{2})^2$

Solution.

This should remind of you of a *perfect square trinomial*:

$$(a+b)^2 = a^2+2ab+b^2$$

Since this expression takes the form of a perfect square trinomial, we can apply the same method as we did in multiplying polynomials. Recall, we are only using the method of a perfect square trinomial.

$$
\begin{array}{rl}
(5\sqrt{7}+\sqrt{2})^2 & \text{Apply perfect square trinomial formula} \\
(5\sqrt{7})^2+2(5\sqrt{7})(\sqrt{2})+(\sqrt{2})^2 & \text{Simplify each term} \\
25\cdot\sqrt{7^2}+10\sqrt{14}+\sqrt{2^2} & \text{Notice, } (\sqrt{7})^2=\sqrt{7^2} \text{ and } (\sqrt{2})^2=\sqrt{2^2} \\
25\cdot 7+10\sqrt{14}+2 & \text{Multiply} \\
175+10\sqrt{14}+2 & \text{Combine like terms} \\
177+10\sqrt{14} & \text{Simplified expression}
\end{array}
$$

Example 10.22

Simplify: $(8-\sqrt{5})(8+\sqrt{5})$

Solution.

This should remind of you of a *difference of two squares*:

$$(a+b)(a-b)=a^2-b^2$$

Since this expressions takes the form of a difference of two squares, we can apply the same method as we did in multiplying polynomials. Recall, we are only using the method of a difference of two squares.

$$\begin{aligned} \left(8-\sqrt{5}\right)\left(8+\sqrt{5}\right) &\quad \text{Apply difference of two squares formula} \\ \left(8\right)^2-\left(\sqrt{5}\right)^2 &\quad \text{Simplify each term} \\ 64-\sqrt{5^2} &\quad \text{Notice, } (\sqrt{5})^2=\sqrt{5^2} \\ 64-5 &\quad \text{Subtract} \\ 59 &\quad \text{Simplified expression} \end{aligned}$$

It's interesting that the original expression contains radicals and the simplified expression contains no radicals. This displays that even though the original expression may contain radicals, in the process of simplifying, we may result in reducing out all radicals.

10.3.5 Simplify quotients with radicals

Division with radicals is very similar to multiplication. If we think about division as reducing fractions, we can reduce the coefficients outside the radicals and reduce the values inside the radicals.

Example 10.23

Simplify: $\dfrac{-3+\sqrt{27}}{3}$

Solution.

We simplify the $\sqrt{27}$ and then try to reduce the fraction.

$$\begin{aligned} \frac{-3+\sqrt{27}}{3} &\quad \text{Rewrite the radicand} \\ \frac{-3+\sqrt{9\cdot 3}}{3} &\quad \text{Apply product rule to the numerator} \\ \frac{-3+3\sqrt{3}}{3} &\quad \text{Factor the numerator} \\ \frac{3(-1+\sqrt{3})}{3} &\quad \text{Reduce the fraction by a factor 3} \\ \frac{\cancel{3}(-1+\sqrt{3})}{\cancel{3}} &\quad \text{Simplify} \\ -1+\sqrt{3} &\quad \text{Simplified expression} \end{aligned}$$

Quotient rule for radicals

If a, b are any two positive real numbers, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

In general, if a, b are any two positive real numbers, then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}},$$

where n is a positive integer and $n \geq 2$.

Example 10.24

Simplify: $\dfrac{\sqrt{44y^6a^4}}{\sqrt{9y^2a^8}}$

Solution.

We apply the quotient rule of radicals and then simplify the radicand:

$$\begin{array}{rl}
\dfrac{\sqrt{44y^6a^4}}{\sqrt{9y^2a^8}} & \text{Apply the quotient rule} \\[2ex]
\sqrt{\dfrac{44y^6a^4}{9y^2a^8}} & \text{Reduce the radicand} \\[2ex]
\sqrt{\dfrac{44\cancelto{4}{y^6}\cancel{a^4}}{9\cancel{y^2}\cancelto{4}{a^8}}} & \text{Simplify} \\[2ex]
\sqrt{\dfrac{44y^4}{9a^4}} & \text{Apply the quotient rule} \\[2ex]
\dfrac{\sqrt{44y^4}}{\sqrt{9a^4}} & \text{Simplify the radicals} \\[2ex]
\dfrac{\sqrt{4\cdot 11\cdot y^4}}{3a^2} & \text{Rewrite} \\[2ex]
\dfrac{2y^2\sqrt{11}}{3a^2} & \text{Simplified expression}
\end{array}$$

Example 10.25

Simplify: $\dfrac{15\sqrt[3]{108}}{20\sqrt[3]{2}}$

Solution.

First we simplify the coefficients, then apply the quotient rule.

$$
\begin{array}{rl}
\dfrac{\cancel{15}^{\,3} \cdot \sqrt[3]{108}}{\cancel{20}^{\,4} \cdot \sqrt[3]{2}} & \text{Simplify coefficients} \\[2ex]
\dfrac{3\sqrt[3]{108}}{4\sqrt[3]{2}} & \text{Apply quotient rule} \\[2ex]
\dfrac{3}{4} \cdot \sqrt[3]{\dfrac{108}{2}} & \text{Reduce the radicand} \\[2ex]
\dfrac{3}{4} \cdot \sqrt[3]{54} & \text{Rewrite the radicand} \\[2ex]
\dfrac{3}{4} \cdot \sqrt[3]{27 \cdot 2} & \text{Apply product rule} \\[2ex]
\dfrac{3}{4} \cdot 3 \cdot \sqrt[3]{2} & \text{Rewrite as one fraction} \\[2ex]
\dfrac{3 \cdot 3\sqrt[3]{2}}{4} & \text{Multiply coefficients} \\[2ex]
\dfrac{9\sqrt[3]{2}}{4} & \text{Simplified expression}
\end{array}
$$

10.3.6 Multiply and Divide Radicals Homework

Simplify.

1. $3\sqrt{5}\cdot -4\sqrt{16}$
2. $\sqrt{12m}\cdot\sqrt{15m}$
3. $\sqrt[3]{4x^3}\cdot\sqrt[3]{2x^4}$
4. $\sqrt{6}(\sqrt{2}+2)$
5. $-5\sqrt{15}(3\sqrt{3}+2)$
6. $5\sqrt{10}(5n+\sqrt{2})$
7. $(2+2\sqrt{2})(-3+\sqrt{2})$
8. $(\sqrt{5}-5)(2\sqrt{5}-1)$
9. $(\sqrt{2a}+2\sqrt{3a})(3\sqrt{2a}+\sqrt{5a})$
10. $(-5-4\sqrt{3})(-3-4\sqrt{3})$
11. $\dfrac{\sqrt{12}}{5\sqrt{100}}$
12. $\dfrac{\sqrt{5}}{4\sqrt{125}}$
13. $\dfrac{\sqrt{10}}{\sqrt{8}}$
14. $\dfrac{2\sqrt{3}}{3\sqrt{4}}$
15. $\dfrac{5x^2}{4\sqrt{9x^4y^8}}$
16. $\dfrac{\sqrt{12p^2}}{\sqrt{3p}}$
17. $\dfrac{3\sqrt[3]{10}}{5\sqrt[3]{27}}$
18. $\dfrac{\sqrt[3]{5}}{4\sqrt[3]{625}}$
19. $\dfrac{5\sqrt[4]{5r^4}}{\sqrt[4]{80r^2}}$
20. $-5\sqrt{10}\cdot\sqrt{15}$
21. $\sqrt{5r^3}\cdot -5\sqrt{10r^2}$
22. $3\sqrt[3]{4a^4}\cdot\sqrt[3]{10a^3}$
23. $\sqrt{10}(\sqrt{5}+\sqrt{2})$
24. $5\sqrt{15}(3\sqrt{3}+2)$
25. $\sqrt{15}(\sqrt{5}-3\sqrt{3v})$
26. $(-2+\sqrt{3})(-5+2\sqrt{3})$
27. $(2\sqrt{3}+\sqrt{5})(5\sqrt{3}+2\sqrt{4})$
28. $(-2\sqrt{2p}+5\sqrt{5})(\sqrt{5p}+\sqrt{5p})$
29. $(5\sqrt{2}-1)(-\sqrt{2m}+5)$
30. $\dfrac{\sqrt{15}}{2\sqrt{4}}$
31. $\dfrac{\sqrt{12}}{\sqrt{3}}$
32. $\dfrac{\sqrt{2}}{3\sqrt{32}}$
33. $\dfrac{4\sqrt{30}}{\sqrt{15}}$
34. $\dfrac{4\sqrt{12xy^{10}}}{5\sqrt{3xy^4}}$
35. $\dfrac{\sqrt{8n^2}}{\sqrt{32n}}$
36. $\dfrac{\sqrt[3]{15}}{\sqrt[3]{64}}$
37. $\dfrac{\sqrt[4]{4}}{2\sqrt[4]{64}}$
38. $\dfrac{4m}{\sqrt[4]{81m^4n^4}}$

10.4 Rationalize denominators

Definition

Rationalizing the denominator is the process for obtaining denominators without radicals.

When given a quotient with radicals, it is common practice to leave an expression without a radical in the denominator. After simplifying an expression, if there is a radical in the denominator, we will rationalize it so that the denominator is left without any radicals. We start by rationalizing denominators with square roots, and then extend this idea to higher roots.

10.4.1 Rationalizing denominators with square roots

Rationalizing the denominator with square roots

To rationalize the denominator with a square root, multiply the numerator and denominator by the exact radical in the denominator, e.g,

$$\frac{1}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

Example 10.26

Simplify: $\dfrac{\sqrt{6}}{\sqrt{5}}$

Solution.

We see the expression is irreducible and that the denominator contains $\sqrt{5}$. We rationalize the denominator so that the denominator is left without radicals.

$\dfrac{\sqrt{6}}{\sqrt{5}}$	Rationalize the denominator
$\dfrac{\sqrt{6}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}}$	Multiply fractions
$\dfrac{\sqrt{6} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}}$	Apply product rule
$\dfrac{\sqrt{30}}{\sqrt{25}}$	Simplify radicals
$\dfrac{\sqrt{30}}{5}$	Simplified expression

Notice, the expression is simplified completely and there are no longer any radicals in the denominator. This is the goal for these problems.

Example 10.27

Simplify: $\dfrac{6\sqrt{14}}{12\sqrt{22}}$

Solution.

We see the expression isn't reduced. We will reduce the fraction by applying the quotient rule, then rationalize the denominator, if needed.

$$\frac{6\sqrt{14}}{12\sqrt{22}} \quad \text{Apply quotient rule}$$

$$\frac{6}{12} \cdot \sqrt{\frac{14}{22}} \quad \text{Reduce fractions}$$

$$\frac{1}{2} \cdot \sqrt{\frac{7}{11}} \quad \text{Rewrite as one fraction}$$

$$\frac{\sqrt{7}}{2\sqrt{11}} \quad \text{Rationalize the denominator}$$

$$\frac{\sqrt{7}}{2\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} \quad \text{Multiply fractions}$$

$$\frac{\sqrt{7} \cdot \sqrt{11}}{2 \cdot \sqrt{11} \cdot \sqrt{11}} \quad \text{Apply product rule}$$

$$\frac{\sqrt{77}}{2 \cdot \sqrt{121}} \quad \text{Simplify radicals}$$

$$\frac{\sqrt{77}}{2 \cdot 11} \quad \text{Simplify radicals}$$

$$\frac{\sqrt{77}}{22} \quad \text{Simplified expression}$$

Example 10.28

Simplify: $\dfrac{\sqrt{3}-9}{2\sqrt{6}}$

Solution.

We see the expression is irreducible and that the denominator contains $\sqrt{6}$. We rationalize the denominator so that the denominator is left without radicals.

$$\frac{\sqrt{3}-9}{2\sqrt{6}} \quad \text{Rationalize the denominator}$$

$$\frac{(\sqrt{3}-9)}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \quad \text{Multiply fractions}$$

$$\frac{\sqrt{6}(\sqrt{3}-9)}{2\sqrt{6} \cdot \sqrt{6}} \quad \text{Distribute and apply product rule}$$

$$\frac{\sqrt{18}-9\sqrt{6}}{2\cdot\sqrt{36}} \qquad \text{Rewrite the radicand 18}$$

$$\frac{\sqrt{9\cdot 2}-9\sqrt{6}}{12} \qquad \text{Simplify radicals}$$

$$\frac{3\sqrt{2}-9\sqrt{6}}{12} \qquad \text{Factor a GCF from the numerator}$$

$$\frac{\cancel{3}(\sqrt{2}-3\sqrt{6})}{\cancel{12}^{4}} \qquad \text{Reduce by a factor of 3}$$

$$\frac{(\sqrt{2}-3\sqrt{6})}{4} \qquad \text{Simplified expression}$$

10.4.2 Rationalizing denominators with higher roots

Radicals with higher roots in the denominators are a bit more challenging. Notice, rationalizing the denominator with square roots works out nicely because we are only trying to obtain a radicand that is a perfect square in the denominator. Here, we are trying to obtain radicands that are perfect cubes or higher in the denominator. Let's try an example.

Example 10.29

Simplify: $\dfrac{4\sqrt[3]{2}}{7\sqrt[3]{25}}$

Solution.

We see the expression is irreducible and that the denominator contains $\sqrt[3]{25}$. We rationalize the denominator so that the denominator is left without radicals. Notice we need a radicand that is a perfect cube in the denominator.

$$\frac{4\sqrt[3]{2}}{7\sqrt[3]{25}} \qquad \text{Rationalize the denominator}$$

$$\frac{4\sqrt[3]{2}}{7\sqrt[3]{25}}\cdot\frac{\sqrt[3]{5}}{\sqrt[3]{5}} \qquad \text{Multiply fractions}$$

$$\frac{4\cdot\sqrt[3]{2}\cdot\sqrt[3]{5}}{7\cdot\sqrt[3]{25}\cdot\sqrt[3]{5}} \qquad \text{Apply product rule}$$

$$\frac{4\sqrt[3]{10}}{7\sqrt[3]{125}} \qquad \text{Simplify radicals}$$

$$\frac{4\sqrt[3]{10}}{7\cdot 5} \qquad \text{Simplify radicals}$$

$$\frac{4\sqrt[3]{10}}{35} \qquad \text{Simplified expression}$$

We choose to multiply by $\sqrt[3]{5}$ because we noticed $\sqrt[3]{25}=\sqrt[3]{5^2}$, and all we needed was an additional factor of 5 to make a perfect cube in the denominator. Since 7 is a coefficient and not a part of the radicand, we do not include it when rationalizing.

Example 10.30

Simplify: $\dfrac{3\sqrt[4]{11}}{\sqrt[4]{2}}$

Solution.

We see the expression is irreducible and that the denominator contains $\sqrt[4]{2}$. We rationalize the denominator so that the denominator is left without radicals. Notice we need a radicand that is a perfect fourth power in the denominator.

$\dfrac{3\sqrt[4]{11}}{\sqrt[4]{2}}$	Rationalize the denominator
$\dfrac{3\sqrt[4]{11}}{\sqrt[4]{2}} \cdot \dfrac{\sqrt[4]{8}}{\sqrt[4]{8}}$	Multiply fractions
$\dfrac{3 \cdot \sqrt[4]{11} \cdot \sqrt[4]{8}}{\sqrt[4]{2} \cdot \sqrt[4]{8}}$	Apply product rule
$\dfrac{3\sqrt[4]{88}}{\sqrt[4]{16}}$	Simplify radicals
$\dfrac{3\sqrt[4]{88}}{2}$	Simplified expression

We choose to multiply by $\sqrt[4]{8}$ because we noticed $\sqrt[4]{2}$, and all we needed was three additional factors of 2 to make a perfect fourth power in the denominator.

10.4.3 Rationalize denominators using the conjugate

There are times where the given denominator is not just one term. Often, in the denominator, we have a difference or sum of two terms in which one or both terms are square roots. In order to rationalize these denominators, we use the idea from a difference of two squares:

$$(a+b)(a-b) = a^2 - b^2$$

Notice, with the difference of two squares, we are left without any outer or inner product terms- just the squares of the first and last terms. Since these denominators take the *form* of a binomial, we have a special name for the factor we use when rationalizing the denominator. The factor is called the *conjugate*.

Rationalize denominators using the conjugate

We rationalize denominators of the type $a \pm \sqrt{b}$ by multiplying the numerator and denominator by their ***conjugates***, e.g.,

$$\frac{1}{a+\sqrt{b}} \cdot \frac{a-\sqrt{b}}{a-\sqrt{b}}$$

The ***conjugate*** for

- $a+\sqrt{b}$ is $a-\sqrt{b}$
- $a-\sqrt{b}$ is $a+\sqrt{b}$

The case is similar for when there is something like $\sqrt{a}\pm\sqrt{b}$ in the denominator.

Putting all these ideas together, let's try an example.

Example 10.31

Simplify: $\dfrac{2}{\sqrt{3}-5}$

Solution.

We notice the difference in the denominator and so we know we will use the conjugate to rationalize the denominator.

$\dfrac{2}{\sqrt{3}-5}$	Rationalize the denominator
$\dfrac{2}{\left(\sqrt{3}-5\right)}\cdot\dfrac{\left(\sqrt{3}+5\right)}{\left(\sqrt{3}+5\right)}$	Multiply fractions
$\dfrac{2\left(\sqrt{3}+5\right)}{\left(\sqrt{3}-5\right)\left(\sqrt{3}+5\right)}$	Distribute and FOIL
$\dfrac{2\sqrt{3}+10}{\sqrt{9}+\cancel{5\sqrt{3}}-\cancel{5\sqrt{3}}-25}$	Simplify
$\dfrac{2\sqrt{3}+10}{3-25}$	Subtract
$\dfrac{2\sqrt{3}+10}{-22}$	Factor a GCF from the numerator
$\dfrac{2\left(\sqrt{3}+5\right)}{-22}$	Reduce by a factor of 2
$\dfrac{\sqrt{3}+5}{-11}$	Rewrite
$-\dfrac{\sqrt{3}+5}{11}$	Simplified expression

Example 10.32

Simplify: $\dfrac{3-\sqrt{5}}{2-\sqrt{3}}$

Solution.

We notice the difference in the denominator and so we know we will use the conjugate to rationalize the denominator.

$$\frac{3-\sqrt{5}}{2-\sqrt{3}} \qquad \text{Rationalize the denominator}$$

$$\frac{(3-\sqrt{5})}{(2-\sqrt{3})} \cdot \frac{(2+\sqrt{3})}{(2+\sqrt{3})} \qquad \text{Multiply fractions}$$

$$\frac{(3-\sqrt{5})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} \qquad \text{FOIL}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{4+\cancel{2\sqrt{3}}-\cancel{2\sqrt{3}}-\sqrt{9}} \qquad \text{Simplify}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{4-3} \qquad \text{Subtract}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{1} \qquad \text{Rewrite}$$

$$6+3\sqrt{3}-2\sqrt{5}-\sqrt{15} \qquad \text{Simplified expression}$$

World Note

During the 5th century BC in India, *Aryabhata* published a treatise on astronomy. His work included a method for finding the square root of numbers that have many digits.

Example 10.33

Simplify: $\dfrac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}}$

Solution.

We notice the sum in the denominator and so we know we will use the conjugate to rationalize the denominator.

$$\frac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}} \qquad \text{Rationalize the denominator}$$

$$\frac{(2\sqrt{5}-3\sqrt{7})}{(5\sqrt{6}+4\sqrt{2})} \cdot \frac{(5\sqrt{6}-4\sqrt{2})}{(5\sqrt{6}-4\sqrt{2})} \qquad \text{Multiply fractions}$$

$$\frac{(2\sqrt{5}-3\sqrt{7})(5\sqrt{6}-4\sqrt{2})}{(5\sqrt{6}+4\sqrt{2})(5\sqrt{6}-4\sqrt{2})} \quad \text{FOIL}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}-12\sqrt{14}}{25\sqrt{36}-\cancel{20\sqrt{12}}+\cancel{20\sqrt{12}}-16\sqrt{4}} \quad \text{Simplify}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}-12\sqrt{14}}{25\cdot 6-16\cdot 2} \quad \text{Subtract}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}-12\sqrt{14}}{118} \quad \text{Simplified expression}$$

10.4.4 Rationalize Denominators Homework

Simplify.

1. $\dfrac{2\sqrt{4}}{3\sqrt{3}}$

2. $\dfrac{\sqrt[3]{5}}{4\sqrt[3]{4}}$

3. $\dfrac{\sqrt{12}}{\sqrt{3}}$

4. $\dfrac{\sqrt{2}}{3\sqrt{5}}$

5. $\dfrac{4\sqrt{3}}{\sqrt{15}}$

6. $\dfrac{4+2\sqrt{3}}{\sqrt{9}}$

7. $\dfrac{4+2\sqrt{3}}{5\sqrt{4}}$

8. $\dfrac{2-5\sqrt{5}}{4\sqrt{13}}$

9. $\dfrac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}$

10. $\dfrac{5}{3\sqrt{5}+\sqrt{2}}$

11. $\dfrac{2}{5+\sqrt{2}}$

12. $\dfrac{3}{4-3\sqrt{3}}$

13. $\dfrac{4}{3+\sqrt{5}}$

14. $-\dfrac{4}{4-4\sqrt{2}}$

15. $\dfrac{1}{1+\sqrt{2}}$

16. $\dfrac{\sqrt{14}-2}{\sqrt{7}-\sqrt{2}}$

17. $\dfrac{\sqrt{ab}-a}{\sqrt{b}-\sqrt{a}}$

18. $\dfrac{a+\sqrt{ab}}{\sqrt{a}+\sqrt{b}}$

19. $\dfrac{2+\sqrt{6}}{2+\sqrt{3}}$

20. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$

21. $\dfrac{6}{3\sqrt{2}-2\sqrt{3}}$

22. $\dfrac{a-b}{a\sqrt{b}-b\sqrt{a}}$

23. $\dfrac{2-\sqrt{5}}{-3+\sqrt{5}}$

24. $\dfrac{-4+\sqrt{3}}{4\sqrt{9}}$

25. $\dfrac{2\sqrt{3}-2}{2\sqrt{16}}$

26. $\dfrac{\sqrt{5}+4}{4\sqrt{17}}$

27. $\dfrac{\sqrt{5}-\sqrt{2}}{3\sqrt{6}}$

28. $\dfrac{5}{\sqrt{3}+4\sqrt{5}}$

29. $\dfrac{5}{2\sqrt{3}-\sqrt{2}}$

30. $\dfrac{4}{\sqrt{2}-2}$

31. $\dfrac{2}{2\sqrt{5}+2\sqrt{3}}$

32. $\dfrac{4}{4\sqrt{3}-\sqrt{5}}$

33. $\dfrac{3+\sqrt{3}}{\sqrt{3}-1}$

34. $\dfrac{2+\sqrt{10}}{\sqrt{2}+\sqrt{5}}$

35. $\dfrac{\sqrt{14}-\sqrt{7}}{\sqrt{14}+\sqrt{7}}$

36. $\dfrac{a+\sqrt{ab}}{\sqrt{a}+\sqrt{b}}$

37. $\dfrac{2\sqrt{5}+\sqrt{3}}{1-\sqrt{3}}$

38. $\dfrac{a-b}{\sqrt{a}+\sqrt{b}}$

39. $\dfrac{ab}{a\sqrt{b}-b\sqrt{a}}$

40. $\dfrac{4\sqrt{2}+3}{3\sqrt{2}+\sqrt{3}}$

41. $\dfrac{-1+\sqrt{5}}{2\sqrt{5}+5\sqrt{2}}$

42. $\dfrac{5\sqrt{2}+\sqrt{3}}{5+5\sqrt{2}}$

43. $\dfrac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$

10.5 Radicals with mixed indices

Knowing that a radical has the same properties as exponents (written as a ratio) allows us to manipulate radicals in new ways. One thing we are allowed to do is reduce, not just the radicand, but the index as well. Let's take a look at a simple example.

Example 10.34

Rewrite $\sqrt[8]{x^6y^2}$ as a reduced radical with root 4.

Solution.

We can rewrite the radical in its rational exponent form, then reduce each exponent fraction.

$\sqrt[8]{x^6y^2}$	Rewrite the root 8 as a rational exponent
$(x^6y^2)^{\frac{1}{8}}$	Multiply exponents
$x^{\frac{6}{8}}y^{\frac{2}{8}}$	Reduce each exponent fraction
$x^{\frac{3}{4}}y^{\frac{1}{4}}$	All exponents have denominator 4, rewrite in radical form
$\sqrt[4]{x^3y}$	Radical in reduced form with root 4

10.5.1 Reduce radicals

Notice we reduced the index by dividing the index and all exponents in the radicand by the same number, e.g., 2 in Example 10.34. If we notice a common factor between the index and all exponents of every factor in the radicand, then we can reduce the radical by dividing by that common factor.

Reduce radicals

If given a radical with root $m \cdot n$ and radicand a^{mp}, then

$$\sqrt[mn]{a^{mp}} = \sqrt[\cancel{m}n]{a^{\cancel{m}p}} = \sqrt[n]{a^p}$$

Example 10.35

Reduce: $\sqrt[24]{a^6b^9c^{15}}$

Solution.

We can rewrite the radical with the root and exponents in the radicand as a product with a common factor, then reduce the radical.

$\sqrt[24]{a^6b^9c^{15}}$	Rewrite root and each exponent as a product with the common factor 3
$\sqrt[3\cdot 8]{a^{3\cdot 2}b^{3\cdot 3}c^{3\cdot 5}}$	Reduce by a common factor of 3
$\sqrt[\cancel{3}\cdot 8]{a^{\cancel{3}\cdot 2}b^{\cancel{3}\cdot 3}c^{\cancel{3}\cdot 5}}$	Simplify
$\sqrt[8]{a^2b^3c^5}$	Radical in reduced form with root 8

We can use the same process even if there are coefficients in the radicand. We just have to rewrite the coefficient with an exponent that includes the common factor of the exponents, and then reduce the radical as usual.

Example 10.36

Reduce: $\sqrt[9]{8m^6n^3}$

Solution.

First, we'll need to rewrite the coefficient 8 with an exponent that includes the common factor of the exponents. Then we can reduce the radical as usual.

$$\begin{aligned}
\sqrt[9]{8m^6n^3} &\quad \text{Rewrite coefficient 8 with an exponent including the common factor 3}\\
\sqrt[9]{2^{3}m^6n^3} &\quad \text{Rewrite root and each exponent as a product with the common factor 3}\\
\sqrt[3\cdot 3]{2^{3\cdot 1}m^{3\cdot 2}n^{3\cdot 1}} &\quad \text{Reduce by a common factor of 3}\\
\sqrt[\cancel{3}\cdot 3]{2^{\cancel{3}\cdot 1}m^{\cancel{3}\cdot 2}n^{\cancel{3}\cdot 1}} &\quad \text{Simplify}\\
\sqrt[3]{2m^2n} &\quad \text{Radical in reduced form with root 3}
\end{aligned}$$

10.5.2 Multiply radicals with different indices

We can apply the method of reducing radicals to multiply radicals with different indices. Let's consider an example using rational exponents, then identify a pattern.

Example 10.37

Multiply: $\sqrt[3]{ab^2}\cdot\sqrt[4]{a^2b}$

Solution.

We can rewrite the radicals in its rational exponent form, find a common denominator, then reduce each exponent fraction.

$$\begin{aligned}
\sqrt[3]{ab^2}\,\sqrt[4]{a^2b} &\quad \text{Rewrite as rational exponents}\\
(ab^2)^{\frac{1}{3}}(a^2b)^{\frac{1}{4}} &\quad \text{Multiply exponents}\\
a^{\frac{1}{3}}b^{\frac{2}{3}}a^{\frac{2}{4}}b^{\frac{1}{4}} &\quad \text{Rewrite each exponent with common denominator 12}\\
a^{\frac{4}{12}}b^{\frac{8}{12}}a^{\frac{6}{12}}b^{\frac{3}{12}} &\quad \text{Rewrite in radical form with index 12}\\
\sqrt[12]{a^4\cdot b^8\cdot a^6\cdot b^3} &\quad \text{Add exponents with same base}\\
\sqrt[12]{a^{10}b^{11}} &\quad \text{Product with common root 12}
\end{aligned}$$

To multiply radicals with different indices, we need to find a common denominator, which is the lowest common multiple (LCM) between the roots. Once we obtain the LCM, we can multiply each root and exponent in the radicand to obtain the LCM, and rewrite as one radical.

Multiply radicals with different indices

Let n, p, m be positive nonzero integers, and the lowest common multiple be m, i.e., $LCM(n, p) = m$, then

$$\sqrt[n]{a}\cdot\sqrt[p]{b} = \sqrt[m]{a^r}\cdot\sqrt[m]{b^t} = \sqrt[m]{a^rb^t},$$

where the exponents $r = \frac{m}{n}$ and $t = \frac{m}{p}$.

Example 10.38

Multiply: $\sqrt[4]{a^2b^3}\cdot\sqrt[6]{a^2b}$

Solution.

Let's find the $LCM(4,6)$ and rewrite each radical with the LCM. Then write as one radical.

$\sqrt[4]{a^2b^3}\cdot\sqrt[6]{a^2b}$	Rewrite radicals with LCM 12
$\sqrt[3\cdot 4]{a^{3\cdot 2}b^{3\cdot 3}}\cdot\sqrt[2\cdot 6]{a^{2\cdot 2}b^{2\cdot 1}}$	Multiply 3 through first radical and multiply 2 through second radical
$\sqrt[12]{a^6b^9}\cdot\sqrt[12]{a^4b^2}$	Simplify and write as one radical with root 12
$\sqrt[12]{a^6b^9\cdot a^4b^2}$	Add exponents with same base
$\sqrt[12]{a^{10}b^{11}}$	Product with common root 12

Example 10.39

Multiply: $\sqrt[5]{x^3y^4}\cdot\sqrt[3]{x^2y}$

Solution.

Let's find the $LCM(3,5)$ and rewrite each radical with the LCM. Then write as one radical.

$\sqrt[5]{x^3y^4}\cdot\sqrt[3]{x^2y}$	Rewrite radicals with LCM 15
$\sqrt[3\cdot 5]{x^{3\cdot 3}y^{3\cdot 4}}\cdot\sqrt[5\cdot 3]{x^{5\cdot 2}y^{5\cdot 1}}$	Multiply 3 through first radical and multiply 5 through second radical
$\sqrt[15]{x^9y^{12}}\cdot\sqrt[15]{x^{10}y^5}$	Simplify and write as one radical with root 15
$\sqrt[15]{x^9y^{12}\cdot x^{10}y^5}$	Add exponents with same base
$\sqrt[15]{x^{19}y^{17}}$	Simplify by extracting out one factor of x and y
$xy\sqrt[15]{x^4y^2}$	Product with common root 15 and extracted factors x and y

Example 10.40

Multiply: $\sqrt{3x(y+z)}\cdot\sqrt[3]{9x(y+z)^2}$

Solution.

Let's find the $LCM(2,3)$ and rewrite each radical with the LCM. Then write as one radical. Note, even though there is a binomials in each radicand, the method stays the same. Recall, methods never change, only problems.

$\sqrt{3x(y+z)}\cdot\sqrt[3]{9x(y+z)^2}$	Rewrite radicals with LCM 6
$\sqrt[3\cdot 2]{3^{3\cdot 1}x^{3\cdot 1}(y+z)^{3\cdot 1}}\cdot\sqrt[2\cdot 3]{3^{2\cdot 2}x^{2\cdot 1}(y+z)^{2\cdot 2}}$	Multiply 3 through first radical and multiply 2 through second radical
$\sqrt[6]{3^3x^3(y+z)^3}\cdot\sqrt[6]{3^4x^2(y+z)^4}$	Simplify and write as one radical with root 6
$\sqrt[6]{3^3x^3(y+z)^3\cdot 3^4x^2(y+z)^4}$	Add exponents with same base
$\sqrt[6]{3^7x^5(y+z)^7}$	Simplify by reducing out one factor of 3 and $(y+z)$
$3(y+z)\sqrt[6]{3x^5(y+z)}$	Product with common root 6 and extracted factors 3 and $(y+z)$

World Note

Originally, the radical was just a check mark with the rest of the radical expression in parenthesis. In 1637, Rene Descartes was the first to put a line over the entire radical expression.

10.5.3 Divide radicals with different indices

Luckily, the same process is used for dividing radicals with mixed indices as we used multiplying radicals with mixed indices. Since the final expression cannot have radicals in the denominator, then there may be an additional step of rationalizing the denominator.

Example 10.41

Divide: $\dfrac{\sqrt[6]{x^4y^3z^2}}{\sqrt[8]{x^7y^2z}}$

Solution.

Let's find the $LCM(6,8)$ and rewrite each radical with the LCM. Then write as one radical. Note, even though we are simplifying a quotient, we still rationalize the denominator when necessary.

$\dfrac{\sqrt[6]{x^4y^3z^2}}{\sqrt[8]{x^7y^2z}}$	Rewrite radicals with LCM 24
$\dfrac{\sqrt[4\cdot 6]{x^{4\cdot 4}y^{4\cdot 3}z^{4\cdot 2}}}{\sqrt[3\cdot 8]{x^{3\cdot 7}y^{3\cdot 2}z^{3\cdot 1}}}$	Multiply 4 through numerator radical and multiply 3 through denominator radical
$\dfrac{\sqrt[24]{x^{16}y^{12}z^8}}{\sqrt[24]{x^{21}y^6z^3}}$	Simplify and write as one radical with root 24
$\sqrt[24]{\dfrac{x^{16}y^{12}z^8}{x^{21}y^6z^3}}$	Reduce factors with same base
$\sqrt[24]{\dfrac{y^6z^5}{x^5}}$	Rationalize the denominator
$\dfrac{\sqrt[24]{y^6z^5}}{\sqrt[24]{x^5}}\cdot\dfrac{\sqrt[24]{x^{19}}}{\sqrt[24]{x^{19}}}$	Multiply numerator and denominator by $\sqrt[24]{x^{19}}$
$\dfrac{\sqrt[24]{x^{19}y^6z^5}}{\sqrt[24]{x^{24}}}$	Simplify
$\dfrac{\sqrt[24]{x^{19}y^6z^5}}{x}$	Quotient with common root 24 and rationalized denominator

10.5.4 Radicals with Mixed Indices Homework

Reduce the following radicals.

1. $\sqrt[8]{16x^4y^6}$

2. $\sqrt[12]{64x^4y^6z^8}$

3. $\sqrt[6]{\dfrac{16x^2}{9y^4}}$

4. $\sqrt[12]{x^6y^9}$

5. $\sqrt[8]{x^6y^4z^2}$

6. $\sqrt[9]{8x^3y^6}$

7. $\sqrt[4]{9x^2y^6}$

8. $\sqrt[4]{\dfrac{25x^3}{16x^5}}$

9. $\sqrt[15]{x^9y^{12}z^6}$

10. $\sqrt[10]{64x^8y^4}$

11. $\sqrt[4]{25y^2}$

12. $\sqrt[16]{81x^8y^{12}}$

Multiply or divide and simplify completely.

13. $\sqrt[3]{5} \cdot \sqrt{6}$

14. $\sqrt{x} \cdot \sqrt[3]{7y}$

15. $\sqrt{x} \cdot \sqrt[3]{x-2}$

16. $\sqrt[5]{x^2y} \cdot \sqrt{xy}$

17. $\sqrt[4]{xy^2} \cdot \sqrt[3]{x^2y}$

18. $\sqrt[4]{a^2bc^2} \cdot \sqrt[5]{a^2b^3c}$

19. $\sqrt{a} \cdot \sqrt[4]{a^3}$

20. $\sqrt[5]{b^2} \cdot \sqrt{b^3}$

21. $\sqrt{xy^3} \cdot \sqrt[3]{x^2y}$

22. $\sqrt[4]{9ab^3} \cdot \sqrt{3a^4b}$

23. $\sqrt[3]{3xy^2z} \cdot \sqrt[4]{9x^3yz^2}$

24. $\sqrt{27a^5(b+1)} \cdot \sqrt[3]{81a(b+1)^4}$

25. $\dfrac{\sqrt[3]{a^2}}{\sqrt[4]{a}}$

26. $\dfrac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$

27. $\dfrac{\sqrt{ab^3c}}{\sqrt[5]{a^2b^3c^{-1}}}$

28. $\dfrac{\sqrt[4]{(3x-1)^3}}{\sqrt[5]{(3x-1)^3}}$

29. $\dfrac{\sqrt[3]{(2x+1)^2}}{\sqrt[5]{(2x+1)^2}}$

30. $\sqrt[3]{7} \cdot \sqrt[4]{5}$

31. $\sqrt[3]{y} \cdot \sqrt[5]{3z}$

32. $\sqrt[4]{3x} \cdot \sqrt{y+4}$

33. $\sqrt{ab} \cdot \sqrt[5]{2a^2b^2}$

34. $\sqrt[5]{a^2b^3} \cdot \sqrt[4]{a^2b}$

35. $\sqrt[6]{x^2yz^3} \cdot \sqrt[5]{x^2yz^2}$

36. $\sqrt[3]{x^2} \cdot \sqrt[6]{x^5}$

37. $\sqrt[4]{a^3} \cdot \sqrt[3]{a^2}$

38. $\sqrt[5]{a^3b} \cdot \sqrt{ab}$

39. $\sqrt{2x^3y^3} \cdot \sqrt[3]{4xy^2}$

40. $\sqrt{a^4b^3c^4} \cdot \sqrt[3]{ab^2c}$

41. $\sqrt{8x(y+z)^5} \cdot \sqrt[3]{4x^2(y+z)^2}$

42. $\dfrac{\sqrt[3]{x^2}}{\sqrt[5]{x}}$

43. $\dfrac{\sqrt[5]{a^4b^2}}{\sqrt[3]{ab^2}}$

44. $\dfrac{\sqrt[5]{x^3y^4z^9}}{\sqrt{xy^{-2}z}}$

45. $\dfrac{\sqrt[3]{(2+5x)^2}}{\sqrt[4]{(2+5x)}}$

46. $\dfrac{\sqrt[4]{(5-3x)^3}}{\sqrt[3]{(5-3x)^2}}$

10.6 Radical equations

Here we look at equations with radicals. As you might expect, to clear a radical we can raise both sides to an exponent. Recall, the roots of radicals can be thought of reversing an exponent. Hence, to reverse a radical, we will use exponents.

Solving radical equations

If $x \geq 0$ and $a \geq 0$, then

$$\sqrt{x} = a \text{ if and only if } x = a^2$$

In general, if $x \geq 0$ and a is a real number, then

$$\sqrt[n]{x} = a \text{ if and only if } x = a^n$$

We assume in this chapter that all variables are greater than or equal to zero.

Recall, if n is even, then x must be greater than or equal to zero. If n is odd, then x can be any real number.

We can apply the following method to solve equations with radicals.

Steps for solving radical equations

Step 1. Isolate the radical.

Step 2. Raise both sides of the equation to the power of the root (index).

Step 3. Solve the equation as usual.

Step 4. Verify the solution(s). (*Recall, we will omit any extraneous solutions.*)

10.6.1 Radical equations with square roots

Example 10.42

Solve for x: $\sqrt{7x+2} = 4$

Solution.

Step 1. Isolate the radical. Notice the radical is already isolated for us on the left, with no coefficients:

$$\sqrt{7x+2} = 4$$

Step 2. Raise both sides of the equation to the power of the root (index).

$$\begin{aligned} \sqrt{7x+2} &= 4 && \text{Raise each side to the power of 2} \\ \left(\sqrt{7x+2}\right)^2 &= 4^2 && \text{Evaluate} \\ 7x+2 &= 16 \end{aligned}$$

Step 3. Solve the equation as usual.

$$\begin{aligned} 7x+2 &= 16 && \text{Isolate the variable term} \\ 7x &= 14 && \text{Solve for } x \\ x &= 2 && \text{Solution} \end{aligned}$$

Step 4. Verify the solution(s). (*Recall, we will omit any extraneous solutions.*)

$$\begin{aligned} \sqrt{7x+2} &\stackrel{?}{=} 4 \\ \sqrt{7(2)+2} &\stackrel{?}{=} 4 \\ \sqrt{16} &\stackrel{?}{=} 4 \\ 4 &= 4 \end{aligned}$$

Thus, $x = 2$ is, in fact, a solution.

Example 10.43

Solve for x: $\sqrt{x+5} = -1$

Solution.

Step 1. Isolate the radical. Notice the radical is already isolated for us on the left, with no coefficients:

$$\sqrt{x+5} = -1$$

Step 2. Raise both sides of the equation to the power of the root (index).

$$\begin{aligned} \sqrt{x+5} &= -1 && \text{Raise each side to the power of 2} \\ \left(\sqrt{x+5}\right)^2 &= (-1)^2 && \text{Evaluate} \\ x+5 &= 1 \end{aligned}$$

Step 3. Solve the equation as usual.

$$\begin{aligned} x+5 &= 1 && \text{Solve for } x \\ x &= -4 && \text{Solution} \end{aligned}$$

Step 4. Verify the solution(s). (*Recall, we will omit any extraneous solutions.*)

$$\begin{aligned} \sqrt{x+5} &\stackrel{?}{=} -1 \\ \sqrt{(-4)+5} &\stackrel{?}{=} -1 \\ \sqrt{1} &\stackrel{?}{=} -1 \\ 1 &\neq -1 \end{aligned}$$

Oh no! When verifying the solution, we obtained a false statement. Thus, this equation has no solution and $x = -4$ is an extraneous solution.

10.6.2 Isolate the radical

Example 10.44

Solve for x: $x + \sqrt{4x+1} = 5$

Solution.

Step 1. Isolate the radical. Let's isolate the radical on the left by moving the x to the right side.

$$\begin{aligned} x + \sqrt{4x+1} &= 5 \\ \sqrt{4x+1} &= 5 - x \end{aligned}$$

Step 2. Raise both sides of the equation to the power of the root (index).

$$\begin{aligned} \sqrt{4x+1} &= 5 - x && \text{Raise each side to the power of 2} \\ \left(\sqrt{4x+1}\right)^2 &= (5-x)^2 && \text{Evaluate} \\ 4x + 1 &= 25 - 10x + x^2 \end{aligned}$$

Step 3. Solve the equation as usual.

$$\begin{aligned} 4x + 1 &= 25 - 10x + x^2 && \text{Notice the } x^2 \text{ term; solve by factoring} \\ x^2 - 14x + 24 &= 0 && \text{Factor} \\ (x-12)(x-2) &= 0 && \text{Apply zero product rule} \\ x - 12 = 0 \quad \text{or} \quad x - 2 &= 0 && \text{Solve} \\ x = 12 \quad \text{or} \quad x &= 2 && \text{Solutions} \end{aligned}$$

Step 4. Verify the solution(s). (*Recall, we will omit any extraneous solutions.*)

$$\begin{aligned} 12 + \sqrt{4(12)+1} &\stackrel{?}{=} 5 & \qquad 2 + \sqrt{4(2)+1} &\stackrel{?}{=} 5 \\ 12 + \sqrt{49} &\stackrel{?}{=} 5 & 2 + \sqrt{9} &\stackrel{?}{=} 5 \\ 12 + 7 &\stackrel{?}{=} 5 & 2 + 3 &\stackrel{?}{=} 5 \\ 19 &\neq 5 & 5 &= 5 \end{aligned}$$

Since $x = 12$ gives a false statement, then $x = 12$ is an extraneous solution. Thus, $x = 2$ is, in fact, the solution.

World Note

The Babylonians were the first known culture to solve quadratics in radicals as early as 2000 BC.

10.6.3 Radical equations with two square roots

Example 10.45

Solve for x: $\sqrt{2x+1} - \sqrt{x} = 1$

Solution.

Step 1. Isolate the radical. Since there are two radicals in the equation, we will isolate only one of them.

$$\begin{aligned} \sqrt{2x+1} - \sqrt{x} &= 1 \\ \sqrt{2x+1} &= 1 + \sqrt{x} \end{aligned}$$

Step 2. Raise both sides of the equation to the power of the root (index).

$$\begin{aligned} \sqrt{2x+1} &= 1+\sqrt{x} && \text{Raise each side to the power of 2} \\ \left(\sqrt{2x+1}\right)^2 &= \left(1+\sqrt{x}\right)^2 && \text{Evaluate} \\ 2x+1 &= 1+2\sqrt{x}+x \end{aligned}$$

Notice there is $\sqrt{x}$ that still remains in the equation even after squaring each side. Hence, we should repeat steps 1 and 2 again to obtain an equation without radicals.

$$\begin{aligned} 2x+1 &= 1+2\sqrt{x}+x && \text{Isolate the radical} \\ x &= 2\sqrt{x} \\ \frac{x}{2} &= \sqrt{x} && \text{Raise each side to the power of 2} \\ \left(\frac{x}{2}\right)^2 &= \left(\sqrt{x}\right)^2 && \text{Evaluate} \\ \frac{x^2}{4} &= x \end{aligned}$$

Step 3. Solve the equation as usual.

$$\begin{aligned} \frac{x^2}{4} &= x && \text{Notice the } x^2 \text{ term; solve by factoring} \\ \frac{x^2}{4} - x &= 0 && \text{Multiply each term by LCD 4} \\ 4 \cdot \frac{x^2}{4} - 4 \cdot x &= 4 \cdot 0 && \text{Simplify} \end{aligned}$$

$$\begin{aligned} x^2 - 4x &= 0 && \text{Factor} \\ x(x-4) &= 0 && \text{Apply zero product rule} \\ x = 0 \quad \text{or} \quad x-4 &= 0 && \text{Solve} \\ x = 0 \quad \text{or} \quad x &= 4 && \text{Solutions} \end{aligned}$$

Step 4. Verify the solution(s). (*Recall, we will omit any extraneous solutions.*)

$$\begin{aligned} \sqrt{2(0)+1} - \sqrt{0} &\stackrel{?}{=} 1 & \qquad \sqrt{2(4)+1} - \sqrt{4} &\stackrel{?}{=} 1 \\ \sqrt{1} &\stackrel{?}{=} 1 & \sqrt{9} - 2 &\stackrel{?}{=} 1 \\ 1 &= 1 & 3 - 2 &\stackrel{?}{=} 1 \\ 1 &= 1 & 1 &= 1 \end{aligned}$$

Since $x = 0$ and $x = 4$ both give true statements, then $x = 0$ and $x = 4$ are, in fact, the solutions.

10.6.4 Radical equations with higher roots

Example 10.46

Solve for n: $\sqrt[3]{n-1} = -4$

Solution.

Step 1. Isolate the radical. Notice the radical is already isolated for us on the left, with no coefficients:

$$\sqrt[3]{n-1} = -4$$

Step 2. Raise both sides of the equation to the power of the root (index). Notice the root here is 3; hence, we will raise each side to the third power.

$$\begin{aligned} \sqrt[3]{n-1} &= -4 && \text{Raise each side to the power of 3} \\ \left(\sqrt[3]{n-1}\right)^3 &= (-4)^3 && \text{Evaluate} \\ n-1 &= -64 \end{aligned}$$

Step 3. Solve the equation as usual.

$$\begin{aligned} n-1 &= -64 && \text{Isolate the variable term} \\ n &= -63 && \text{Solution} \end{aligned}$$

Step 4. Verify the solution(s). (*Recall, we will omit any extraneous solutions.*)

$$\begin{aligned} \sqrt[3]{n-1} &\stackrel{?}{=} -4 \\ \sqrt[3]{-63-1} &\stackrel{?}{=} -4 \\ \sqrt[3]{-64} &\stackrel{?}{=} -4 \\ -4 &= -4 \end{aligned}$$

Thus, $n = -63$ is, in fact, a solution.

10.6.5 Applications with radical functions

Example 10.47

A person's Body Mass Index (BMI) is a measure of body fat based on height and weight. If a person's BMI is above 25 and below 30, he/she is classified as overweight. A person's height in terms of his/her weight in pounds, w, and body mass index (BMI), b, is given by

$$H(w) = \sqrt{\frac{703w}{b}}$$

a) How tall is a person weighing 250 pounds and has a BMI of 25? *Round your answer to one decimal place.*

b) If a person is 71 inches tall and has a BMI of 25, what is the person's weight? *Round your answer to one decimal place.*

Solution.

We apply the formula to answer both parts. Since this is a function with a square root, we use the techniques from above to solve.

a) Since the person weighs 250 pounds and has BMI 25, then $w = 250$ and $b = 25$. Let's plug-n-chug these in to find the person's height in inches.

$$\begin{aligned} H(w) &= \sqrt{\frac{703w}{b}} \\ H(250) &= \sqrt{\frac{703(250)}{25}} \\ H(250) &= \sqrt{7030} \\ H(250) &\approx 83.5 \end{aligned}$$

Thus, a person whose weight is 250 pounds with a BMI of 25 is about 83.5 inches tall, which is nearly 7 feet tall!

b) Since the person is 71 inches tall and has BMI 25, then $H = 71$ and $b = 25$. Let's plug-n-chug this into the function and solve for w, the person's weight in pounds.

$$\begin{aligned} H(w) &= \sqrt{\frac{703w}{b}} \\ 71 &= \sqrt{\frac{703w}{25}} \\ 71^2 &= \frac{703w}{25} \\ 71^2 \cdot 25 &= 703w \\ \frac{71^2 \cdot 25}{703} &= w \\ w &\approx 179.3 \end{aligned}$$

Thus, a person whose height is 71 inches with a BMI of 25, weighs about 179.3 pounds.

Example 10.48

The time it takes for a pendulum to swing back and forth one time can be represented by the function

$$S(x) = 2\pi\sqrt{\frac{x}{32}}$$

where $S(x)$ is the time in seconds and x is the length of the pendulum in feet.

a) How many seconds will take for a 7-foot pendulum to swing back and forth one time? *Round your answer to one decimal place.*

b) If it takes 4 seconds for a pendulum to swing back and forth one time, what is the length of the pendulum? *Round your answer to one decimal place.*

Solution.

We apply the formula to answer both parts. Since this is a function with a square root, we use the techniques from above to solve.

a) Since it is given that the pendulum is 7 feet, then this implies $x = 7$. Let's plug-n-chug $x = 7$ into S to obtain the time it takes for the pendulum to swing back and forth one time.

$$\begin{aligned} S(x) &= 2\pi\sqrt{\frac{x}{32}} \\ S(7) &= 2\pi\sqrt{\frac{7}{32}} \\ S(7) &\approx 2.9 \end{aligned}$$

Thus, it will take about 2.9 seconds for a 7-foot pendulum to swing back and forth one time.

b) If we are given that it takes 4 seconds for a pendulum to swing back and forth one time, then this means $S = 4$. Let's plug-n-chug this into the function to find x, the length of the pendulum in feet.

$$\begin{aligned} S(x) &= 2\pi\sqrt{\frac{x}{32}} \\ 4 &= 2\pi\sqrt{\frac{x}{32}} \\ \frac{4}{2\pi} &= \sqrt{\frac{x}{32}} \\ \left(\frac{4}{2\pi}\right)^2 &= \frac{x}{32} \\ 32 \cdot \frac{16}{4\pi^2} &= x \\ x &\approx 13.0 \end{aligned}$$

Thus, a 13-foot pendulum will take 4 seconds to swing back and forth one time.

10.6.6 Radical Equations Homework

Solve. *Be sure to verify all solutions.*

1. $\sqrt{2x+3}-3=0$

2. $\sqrt{6x-5}-x=0$

3. $3+x=\sqrt{6x+13}$

4. $\sqrt{3-3x}-1=2x$

5. $\sqrt{4x+5}-\sqrt{x+4}=2$

6. $\sqrt{2x+4}-\sqrt{x+3}=1$

7. $\sqrt{2x+6}-\sqrt{x+4}=1$

8. $\sqrt{6-2x}-\sqrt{2x+3}=3$

9. $\sqrt{5x+1}-4=0$

10. $\sqrt{x+2}-\sqrt{x}=2$

11. $x-1=\sqrt{7-x}$

12. $\sqrt{2x+2}=3+\sqrt{2x-1}$

13. $\sqrt{3x+4}-\sqrt{x+2}=2$

14. $\sqrt{7x+2}-\sqrt{3x+6}=6$

15. $\sqrt{4x-3}-\sqrt{3x+1}=1$

16. $\sqrt{2-3x}-\sqrt{3x+7}=3$

17. A person's Body Mass Index (BMI) is a measure of body fat based on height and weight. If a person's BMI is above 25 and below 30, he/she is classified as overweight. A person's height in terms of his/her weight in pounds, w, and body mass index (BMI), b, is given by

$$H(w)=\sqrt{\frac{703w}{b}}$$

a) How tall is a person if he/she weighs 225 pounds and has a BMI of 25? *Round your answer to one decimal place.*

b) If a person is 64 inches tall and has a BMI of 25, what is his/her weight? *Round your answer to one decimal place.*

18. The time it takes for a pendulum to swing back and forth one time can be represented by the function

$$S(x)=2\pi\sqrt{\frac{x}{32}}$$

where $S(x)$ is the time in seconds and x is the length of the pendulum in feet.

a) How many seconds will take for a 5-foot pendulum to swing back and forth one time? *Round your answer to one decimal place.*

b) If it takes 2 seconds for a pendulum to swing back and forth one time, what is the length of the pendulum? *Round your answer to one decimal place.*

10.7 Solving with rational exponents

In this section, we take solving with radicals one step further and apply radicals and rational exponents to solve equations with exponents. Since radicals have some restrictions on the radicand, we will also have some restrictions here when applying a rational exponents in the solving process.

10.7.1 The odd root property

Let's start with the case that we take an odd root of an equation.

The odd root property

If $x^n = p$, where n is odd, then $x = \sqrt[n]{p}$. Note, the radicand can be any real number, i.e., p is any number in $(-\infty, \infty)$.

Example 10.49

Solve: $x^5 = 32$

Solution.

We can easily apply the odd root property to solve for x.

$$\begin{aligned} x^5 &= 32 && \text{Apply odd root property} \\ \sqrt[5]{x^5} &= \sqrt[5]{32} && \text{Simplify} \\ x &= 2 && \text{Solution} \end{aligned}$$

Example 10.50

Solve: $4r^3 - 2 = 106$

Solution.

We can easily apply the odd root property to solve for r.

$$\begin{aligned} 4r^3 - 2 &= 106 && \text{Isolate the variable term} \\ 4r^3 &= 108 && \text{Isolate } r^3 \\ r^3 &= 27 && \text{Apply odd root property} \\ \sqrt[3]{r^3} &= \sqrt[3]{27} && \text{Simplify} \\ r &= 3 && \text{Solution} \end{aligned}$$

10.7.2 The even root property

With even roots, we have the restriction on the radicand where the radicand is required to be non-negative here. We discussed this in the previous section, e.g., $\sqrt{-4}$ is not a real number. We continue this restriction when taking even roots of an equation.

The even root property

If $x^n = p$, where n is even, then $x = \sqrt[n]{p}$ or $x = -\sqrt[n]{p}$ or we can write $x = \pm\sqrt[n]{p}$. Note, the radicand can be any real non-negative number, i.e., $p \geq 0$.

Example 10.51

Solve: $x^4 = 16$

Solution.

We can easily apply the even root property to solve for x.

$$\begin{aligned} x^4 &= 16 && \text{Apply even root property} \\ \sqrt[4]{x^4} &= \sqrt[4]{16} && \text{Simplify} \\ |x| &= \pm 2 && \\ x &= \pm 2 && \text{Solution} \end{aligned}$$

Notice, it wasn't given that $x \geq 0$. Hence, we cannot assume it is, so we put absolute value around x. Once we verify the solution(s), then we can remove the absolute value around x.

World Note

In 1545, French mathematician *Gerolamo Cardano* published his book *The Great Art, or the Rules of Algebra*, which included the solution to an equation with a fourth power, but it was considered absurd by many to take a quantity to the fourth power because there are only three dimensions!

Example 10.52

Solve: $(2x+4)^2 = 36$ *Find and verify all solutions that satisfy the equation.*

Solution.

We can easily apply the even root property to solve for x.

$$\begin{aligned} (2x+4)^2 &= 36 && \text{Apply even root property} \\ \sqrt{(2x+4)^2} &= \pm\sqrt{36} && \text{Simplify } \sqrt{36} \\ 2x+4 &= \pm 6 && \text{Rewrite into two equations} \\ 2x+4 = 6 \quad \text{or} \quad 2x+4 &= -6 && \text{Isolate the variable term in each equation} \\ 2x = 2 \quad \text{or} \quad 2x &= -10 && \text{Solve each equation} \\ x = 1 \quad \text{or} \quad x &= -5 && \text{Solutions} \end{aligned}$$

We can always verify the solutions by substituting back in $1, -5$ into the original equation:

$$\begin{aligned} (2x+4)^2 &= 36 && \text{Plug-n-chug } x = 1 \\ (2(1)+4)^2 &\stackrel{?}{=} 36 && \text{Simplify each side} \\ (2+4)^2 &\stackrel{?}{=} 36 && \\ 6^2 &\stackrel{?}{=} 36 && \\ 36 &= 36 && \checkmark\text{True} \end{aligned}$$

Let's try the next solution $x = -5$:

$$\begin{aligned} (2x+4)^2 &= 36 && \text{Plug-n-chug } x=-5 \\ (2(-5)+4)^2 &\stackrel{?}{=} 36 && \text{Simplify each side} \\ (-10+4)^2 &\stackrel{?}{=} 36 && \\ (-6)^2 &\stackrel{?}{=} 36 && \\ 36 &= 36 && \checkmark\text{True} \end{aligned}$$

Thus, $1, -5$ are, in fact, solutions to the original equation.

Example 10.53

Solve: $(6x-9)^2 = 45$ *Find and verify all solutions that satisfy the equation.*

Solution.

$$\begin{aligned} (6x-9)^2 &= 45 && \text{Apply even root property} \\ \sqrt{(6x-9)^2} &= \pm\sqrt{45} && \text{Simplify } \sqrt{45} \\ 6x-9 &= \pm 3\sqrt{5} && \text{Isolate the variable term} \\ 6x &= 9 \pm 3\sqrt{5} && \text{Divide both sides by 6} \\ x &= \frac{9 \pm 3\sqrt{5}}{6} && \text{Factor a GCF from numerator} \\ x &= \frac{\cancel{3}(3 \pm \sqrt{5})}{\cancelto{2}{6}} && \text{Simplify} \\ x &= \frac{3 \pm \sqrt{5}}{2} && \text{Solution} \end{aligned}$$

Notice, we didn't split the equation into two different equations and solve. Since $\sqrt{45}$ is an irrational number, we can leave the $\pm$ and solve as usual. *We leave verifying the solutions to the student.*

Example 10.54

Solve $256w^8 + 40 = 41$

Solution.

We have to isolate the variable term first, then we can apply the even root property.

$$\begin{aligned} 256w^8 + 40 &= 41 && \text{Isolate the variable term.} \\ 256w^8 &= 1 && \text{Divide each side by 256} \\ w^8 &= \frac{1}{256} && \text{Apply even root property} \end{aligned}$$

$$
\begin{aligned}
\sqrt[8]{w^8} &= \pm\sqrt[8]{\frac{1}{256}} && \text{Simplify the radicals}\\
|w| &= \pm\frac{1}{2}\\
w &= \pm\frac{1}{2} && \text{Solution}
\end{aligned}
$$

Notice, it wasn't given that $w \geq 0$. Hence, we cannot assume it is and we put absolute value around w. Once we verify the solution(s), then we can remove the absolute value around w.

10.7.3 Solving equations with rational exponents

When exponents are fractions, we convert the rational exponent into a radical expression to solve. Recall, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$. Then we clear the exponent by applying either the even or odd root property and solve as usual.

Steps for solving equations with rational exponents

Given an equation with rational exponents, we can follow the following steps to solve.

Step 1. Rewrite any rational exponents as radicals.

Step 2. Apply the odd or even root property. *Recall, even roots require the radicand to be positive unless otherwise noted.*

Step 3. Raise each side to the power of the root.

Step 4. Solve. *Verify the solutions, especially when there is an even root.*

Example 10.55

Solve: $(4x+1)^{\frac{2}{5}} = 9$ *Assume all variables are positive.*

Solution.

We follow the steps in order to solve the equation with a rational exponent.

Step 1. Rewrite any rational exponents as radicals.

$$
\begin{aligned}
(4x+1)^{\frac{2}{5}} &= 9\\
\left(\sqrt[5]{4x+1}\right)^2 &= 9
\end{aligned}
$$

Step 2. Apply the odd or even root property. *Recall, even roots require the radicand to be positive unless otherwise noted.*

Since we are taking the square root, which is even, then we apply the even root property:

$$
\begin{aligned}
\left(\sqrt[5]{4x+1}\right)^2 &= 9\\
\sqrt[5]{4x+1} &= \pm\sqrt{9}\\
\sqrt[5]{4x+1} &= \pm 3
\end{aligned}
$$

Step 3. Raise each side to the power of the root.

Since the root is 5, then we can raise each side to the fifth power:

$$\begin{aligned}\sqrt[5]{4x+1} &= \pm 3\\ \left(\sqrt[5]{4x+1}\right)^5 &= (\pm 3)^5\\ 4x+1 &= \pm 243\end{aligned}$$

Step 4. Solve. *Verify the solutions, especially when there is an even root.*

$$\begin{aligned}4x+1 &= 243 &\quad\text{or}\quad& 4x+1=-243\\ 4x &= 242 &\quad\text{or}\quad& 4x=-244\\ x &= \frac{242}{4} &\quad\text{or}\quad& x=-61\\ x &= \frac{121}{2} &\quad\text{or}\quad& x=-61\end{aligned}$$

We can verify the solutions. Let's start by verifying that $x = -61$ is a solution.

$$\begin{aligned}\left(\sqrt[5]{4x+1}\right)^2 &= 9\\ \left(\sqrt[5]{4(-61)+1}\right)^2 &\stackrel{?}{=} 9\\ \left(\sqrt[5]{-244+1}\right)^2 &\stackrel{?}{=} 9\\ \left(\sqrt[5]{-243}\right)^2 &\stackrel{?}{=} 9\\ (-3)^2 &\stackrel{?}{=} 9\\ 9 &= 9\checkmark\text{True}\end{aligned}$$

We leave the verification of the second solution to the student.

Thus, the solutions to the equation are $\dfrac{121}{2}, -61$.

Example 10.56

Solve: $(3x-2)^{\frac{3}{4}} = 64$

Solution.

We follow the steps in order to solve the equation with a rational exponent.

Step 1. Rewrite any rational exponents as radicals.

$$\begin{aligned}(3x-2)^{\frac{3}{4}} &= 64\\ \left(\sqrt[4]{3x-2}\right)^3 &= 64\end{aligned}$$

Step 2. Apply the odd or even root property. *Recall, even roots require the radicand to be positive unless otherwise noted.*

Since we are taking the cube root, which is odd, then we apply the odd root property:

$$\begin{aligned}\left(\sqrt[4]{3x-2}\right)^3 &= 64\\ \sqrt[4]{3x-2} &= \sqrt[3]{64}\\ \sqrt[4]{3x-2} &= 4\end{aligned}$$

Step 3. Raise each side to the power of the root.

Since the root is 4, then we can raise each side to the fourth power:

$$\begin{aligned} \sqrt[4]{3x-2} &= 4 \\ \left(\sqrt[4]{3x-2}\right)^4 &= 4^4 \\ 3x-2 &= 256 \end{aligned}$$

Step 4. Solve. *Verify the solutions, especially when there is an even root.*

$$\begin{aligned} 3x-2 &= 256 \\ 3x &= 258 \\ x &= 86 \end{aligned}$$

Since there is an even root in the original equation, we should verify the solution.

$$\begin{aligned} \left(\sqrt[4]{3x-2}\right)^3 &= 64 \\ \left(\sqrt[4]{3({\color{blue}86})-2}\right)^3 &\stackrel{?}{=} 64 \\ \left(\sqrt[4]{258-2}\right)^3 &\stackrel{?}{=} 64 \\ \left(\sqrt[4]{256}\right)^3 &\stackrel{?}{=} 64 \\ (4)^3 &\stackrel{?}{=} 64 \\ 64 &= 64\checkmark\text{True} \end{aligned}$$

Thus, the solution is 86.

When solving equations with rational exponents, it is very helpful to convert the equations into their radical form so we can see which property we need to use and to identify whether we need to verify the solutions due to an even root in the original equation.

10.7.4 Solving with Rational Exponents Homework

Solve.

1. $x^2 = 75$

2. $x^2 + 5 = 13$

3. $3x^2 + 1 = 73$

4. $(x+2)^5 = -243$

5. $(2x+5)^3 - 6 = 21$

6. $(x-1)^{\frac{2}{3}} = 16$

7. $(2-x)^{\frac{3}{2}} = 27$

8. $(2x-3)^{\frac{2}{3}} = 4$

9. $(x+\frac{1}{2})^{-\frac{2}{3}} = 4$

10. $(x-1)^{-\frac{5}{2}} = 32$

11. $(3x-2)^{\frac{4}{5}} = 16$

12. $(4x+2)^{\frac{3}{5}} = -8$

13. $x^3 = -8$

14. $4x^3 - 2 = 106$

15. $(x-4)^2 = 49$

16. $(5x+1)^4 = 16$

17. $(2x+1)^2 + 3 = 21$

18. $(x-1)^{\frac{3}{2}} = 8$

19. $(2x+3)^{\frac{4}{3}} = 16$

20. $(x+3)^{-\frac{1}{3}} = 4$

21. $(x-1)^{-\frac{5}{3}} = 32$

22. $(x+3)^{\frac{3}{2}} = -8$

23. $(2x+3)^{\frac{3}{2}} = 27$

24. $(3-2x)^{\frac{4}{3}} = -81$

10.8 Complex numbers

World Note

When mathematics was first used, the primary purpose was for counting. Thus, they did not originally use negative numbers, zero, fractions, or irrational numbers. However, the ancient Egyptians quickly developed the need for "a part" and so they made up a new type of number, the *ratio* or fraction. The Ancient Greeks did not believe in irrational numbers (people were killed for believing otherwise). The Mayans of Central America later made up the number zero when they found use for it as a placeholder. Ancient Chinese Mathematicians made up negative numbers when they found use for them.

When working with radicals, we often work with radicands which are greater than or equal to zero. What about the case when the radicand is negative, especially with even roots? Previously, we said numbers like $\sqrt{-4}$ were not *real numbers*, but what kind of number is it? In this event, we call numbers that contain square roots of negative numbers ***complex numbers***. Before we get to the complex number, we discuss the ***imaginary unit***.

10.8.1 Imaginary unit

Definition

The ***imaginary unit***, denoted by i, is the number whose square is -1, i.e.,

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

Example 10.57

Simplify $\sqrt{-16}$ using the imaginary unit.

Solution.

$\sqrt{-16}$	Consider the negative as a factor of -1
$\sqrt{-1 \cdot 16}$	Apply the product property of square roots
$\sqrt{-1} \cdot \sqrt{16}$	Evaluate and rewrite $\sqrt{-1}$ as i
$4i$	$\sqrt{-16}$ using the imaginary unit

Example 10.58

Simplify $\sqrt{-24}$ using the imaginary unit.

Solution.

For this example, we use techniques from simplifying radicals in addition to rewriting the radical with the imaginary unit.

$\sqrt{-24}$	Consider the negative as a factor of -1
$\sqrt{-1 \cdot 24}$	Apply the product property of square roots
$\sqrt{-1} \cdot \sqrt{24}$	Simplify $\sqrt{24}$ and rewrite $\sqrt{-1}$ as i
$i \cdot \sqrt{4 \cdot 6}$	Simplify the radical
$2i\sqrt{6}$	$\sqrt{-24}$ using the imaginary unit

Example 10.59

Simplify $(3i)(7i)$.

Solution.

$(3i)(7i)$	Multiply
$21i^2$	Apply the definition and rewrite i^2 as -1
$21(-1)$	Multiply
-21	Result

Note

As a rule of thumb, we always rewrite $\sqrt{-1}$ as i, and i^2 as -1.

10.8.2 Complex numbers

Definition

A ***complex number*** is a number of the form $a + bi$, where a and b are real numbers, and a is called the *real part* of $a + bi$ and bi is called the *imaginary part* of $a + bi$.

Example 10.60

Express $4 + \sqrt{-64}$ as a complex number in the form $a + bi$.

Solution.

$4 + \sqrt{-64}$	Rewrite $\sqrt{-64}$ as factors 64 and -1
$4 + \sqrt{-1 \cdot 64}$	Apply product property of square roots
$4 + \sqrt{-1} \cdot \sqrt{64}$	Simplify the radicals
$4 + 8i$	Complex number

Here, 4 is the *real* part and $8i$ is the *imaginary* part. Together, they make a complex number.

Example 10.61

Express $7 - \sqrt{-18}$ as a complex number in the form $a + bi$.

Solution.

$$\begin{array}{rl}
7-\sqrt{-18} & \text{Rewrite } \sqrt{-18} \text{ as factors 18 and } -1 \\
7-\sqrt{-1\cdot 18} & \text{Apply product property of square roots} \\
7-\sqrt{-1}\cdot\sqrt{18} & \text{Simplify } \sqrt{18} \text{ and rewrite } \sqrt{-1} \text{ as } i \\
7-i\cdot\sqrt{9\cdot 2} & \text{Simplify the radical} \\
7-3i\sqrt{2} & \text{Complex number}
\end{array}$$

Here, 7 is the *real* part and $-3i\sqrt{2}$ is the *imaginary* part. Together, they make a complex number.

10.8.3 Simplify expressions with complex numbers

When simplifying expressions with complex numbers, it is important that we rewrite any radicals that contain $\sqrt{-1}$ or i^2, replacing them by i and -1, respectively. Then we simplify.

Example 10.62

Simplify $\sqrt{-6}\cdot\sqrt{-3}$.

Solution.

We rewrite each factor using the imaginary unit, then apply the operation.

$$\begin{array}{rl}
\sqrt{-6}\cdot\sqrt{-3} & \text{Rewrite the radicals with } i \\
(i\sqrt{6})(i\sqrt{3}) & \text{Multiply} \\
i^2\cdot\sqrt{18} & \text{Rewrite } i^2 \text{ as } -1 \text{ and simplify the } \sqrt{18} \\
-1\cdot\sqrt{9\cdot 2} & \text{Simplify the radical} \\
-1\cdot 3\sqrt{2} & \text{Simplify the } -1\cdot 3 \\
-3\sqrt{2} & \text{Product}
\end{array}$$

Notice, even though we started with imaginary units, our product didn't contain any because of the i^2 term. Recall, every time we see an i^2, we rewrite it as -1, which contains no i.

Example 10.63

Simplify $\dfrac{-15-\sqrt{-200}}{20}$.

Solution.

We rewrite each term using the imaginary unit as needed, then apply the operation.

$$\begin{array}{rl}
\dfrac{-15-\sqrt{-200}}{20} & \text{Rewrite the radical with } i \text{ and as a product of factors} \\
\dfrac{-15-\sqrt{-1\cdot 100\cdot 2}}{20} & \text{Simplify the radical}
\end{array}$$

$$\frac{-15-10i\sqrt{2}}{20} \qquad \text{Factor a GCF from the numerator}$$

$$\frac{5\left(-3-2i\sqrt{2}\right)}{20} \qquad \text{Reduce the fraction by a factor of 5}$$

$$\frac{\cancel{5}\left(-3-2i\sqrt{2}\right)}{\cancel{20}^{\,4}} \qquad \text{Rewrite}$$

$$\frac{-3-2i\sqrt{2}}{4} \qquad \text{Quotient}$$

The answer above will suffice, but if we wanted to rewrite $\dfrac{-3-2i\sqrt{2}}{4}$ as a standard complex number, then we would rewrite the answer as

$$-\frac{3}{4}-\frac{\sqrt{2}}{2}i$$

where $-\dfrac{3}{4}$ is the *real* part and $-\dfrac{\sqrt{2}}{2}i$ is the *imaginary* part.

10.8.4 Simplify expressions with complex numbers by adding, subtracting, & multiplying

We apply arithmetic operations to complex numbers in a way very similar to the way we apply arithmetic operations with expressions that contain variables. We combine like terms, when necessary. In this case, like terms are those with real parts and those with imaginary parts.

Example 10.64

Add: $(2+5i)+(4-7i)$

Solution.

We simplify by combining like terms: combine real parts and combine imaginary parts.

$$\begin{array}{rl} (2+5i)+(4-7i) & \text{Combine like terms} \\ \underbrace{(2+4)}_{\text{real parts}}+\underbrace{(5i-7i)}_{\text{imaginary}} & \text{Simplify} \\ 6-2i & \text{Simplified expression} \end{array}$$

Example 10.65

Subtract: $(4-8i)-(3-5i)$

Solution.

We simplify by combining like terms: combine real parts and combine imaginary parts, but, first, we distribute the subtraction to each term in the parenthesis after the subtraction sign.

$$\begin{aligned}
(4-8i)-(3-5i) &\quad \text{Distribute the negative}\\
4-8i-3+5i &\quad \text{Combine like terms}\\
\underbrace{(4-3)}_{\text{real parts}}+\underbrace{(5i-8i)}_{\text{imaginary}} &\quad \text{Simplify}\\
1-3i &\quad \text{Simplified expression}
\end{aligned}$$

Example 10.66

Simplify: $(5i)-(3+8i)+(-4+7i)$

Solution.

We simplify by combining like terms: combine real parts and combine imaginary parts, but, first, we distribute the subtraction to each term in the parenthesis after the subtraction sign.

$$\begin{aligned}
(5i)-(3+8i)+(-4+7i) &\quad \text{Distribute the negative}\\
5i-3-8i-4+7i &\quad \text{Combine like terms}\\
\underbrace{(-3-4)}_{\text{real parts}}+\underbrace{(5i-8i+7i)}_{\text{imaginary}} &\quad \text{Simplify}\\
-7+4i &\quad \text{Simplified expression}
\end{aligned}$$

Note

Multiplying with complex numbers is similar to multiplying with variables except we rewrite every

$$\sqrt{-1} \quad \text{as} \quad i \quad \textit{and} \quad i^2 \quad \text{as} \quad -1$$

Example 10.67

Simplify: $5i(3i-7)$

Solution.

We multiply as usual applying the same exponent rules.

$$\begin{aligned}
5i(3i-7) &\quad \text{Distribute } 5i\\
15i^2-35i &\quad \text{Rewrite } i^2=-1\\
15(-1)-35i &\quad \text{Simplify}\\
-15-35i &\quad \text{Simplified expression}
\end{aligned}$$

Multiplying expressions where the factors take a binomial form, we an apply the moethod of FOIL, a method we discussed in the exponents and polynomial chapter.

Note

Recall, the FOIL method:

First	-Multiply the first terms in each parenthesis
Outer	-Multiply the outer terms in each parenthesis
Inner	-Multiply the inner terms in each parenthesis
Last	-Multiply the last terms in each parenthesis

Example 10.68

Simplify: $(2-4i)(3+5i)$

Solution.

We multiply this expression using the method of FOIL.

$$\begin{array}{rl} (2-4i)(3+5i) & \text{FOIL} \\ 6+10i-12i-20i^2 & \text{Rewrite } i^2=-1 \\ 6+10i-12i-20(-1) & \text{Simplify} \\ 6+10i-12i+20 & \text{Combine like terms} \\ 26-2i & \text{Simplified expression} \end{array}$$

Example 10.69

Simplify: $(4-5i)^2$

Solution.

We multiply this expression using either the method of FOIL or the perfect square trinomial formula, where $(A-B)^2 = A^2 - 2AB + B^2$. Let's use the perfect square trinomial formula.

$$\begin{array}{rl} (4-5i)^2 & \text{Apply the perfect square trinomial formula} \\ (4)^2-2(4)(5i)+(5i)^2 & \text{Simplify} \\ 16-40i+25i^2 & \text{Rewrite } i^2=-1 \\ 16-40i+25(-1) & \text{Simplify} \\ 16-40i-25 & \text{Combine like terms} \\ -9-40i & \text{Simplified expression} \end{array}$$

Example 10.70

Simplify: $(3i)(6i)(2-3i)$

Solution.

We multiply this expression as usual and with distribution.

$(3i)(6i)(2-3i)$	Multiply first two monomials
$18i^2(2-3i)$	Distribute $18i^2$
$36i^2 - 54i^3$	Rewrite $i^3 = i^2 \cdot i$
$36i^2 - 54i^2 \cdot i$	Rewrite $i^2 = -1$
$36(-1) - 54(-1)i$	Simplify
$-36 + 54i$	Simplified expression

10.8.5 Simplify expressions with complex numbers by applying the conjugate

Dividing with complex numbers is interesting if we have an imaginary part in the denominator. What do we do with an i in the denominator? Let's think about i and its representation: $i = \sqrt{-1}$. If there is an i in the denominator, then there is a square root in the denominator. Hence, we have to rationalize the denominator, but now using complex numbers.

Rationalize denominators with an imaginary part

To rationalize the denominator with only an imaginary part in the denominator, multiply the numerator and denominator by i, e.g,

$$\frac{1}{i} \cdot \frac{i}{i}$$

Example 10.71

Simplify: $\dfrac{7+3i}{-5i}$

Solution.

We see that there is a $-5i$ in the denominator. We can multiply the numerator and denominator by i to rewrite the denominator without i, i.e., without a square root.

$\dfrac{7+3i}{-5i}$	Multiply numerator and denominator by i
$\dfrac{(7+3i)}{-5i} \cdot \dfrac{i}{i}$	Distribute i in numerator
$\dfrac{7i+3i^2}{-5i^2}$	Rewrite $i^2 = -1$
$\dfrac{7i+3(-1)}{-5(-1)}$	Simplify
$\dfrac{7i-3}{5}$	Simplified expression

There are times where the given denominator is not just the imaginary part. Often, in the denominator, we have a complex number. In order to rationalize these denominators, we use the *conjugate*.

Rationalize denominators with a complex number using the conjugate

We rationalize denominators with complex numbers of the type $a \pm bi$ by multiplying the numerator and denominator by their ***conjugates***, e.g.,

$$\frac{1}{a+bi} \cdot \frac{a-bi}{a-bi}$$

The ***conjugate*** for

- $a + bi$ is $a - bi$
- $a - bi$ is $a + bi$

Example 10.72

Simplify: $\dfrac{2-6i}{4+8i}$

Solution.

We see that there is a $4+8i$ in the denominator. We can multiply the numerator and denominator by $4-8i$ to rewrite the denominator without i, i.e., without a square root.

$$\begin{aligned}
&\frac{2-6i}{4+8i} && \text{Multiply numerator and denominator by conjugate}\\
&\frac{2-6i}{4+8i}\cdot\frac{4-8i}{4-8i} && \text{Multiply numerator and denominator}\\
&\frac{8-16i-24i+48i^2}{16-64i^2} && \text{Rewrite } i^2=-1\\
&\frac{8-16i-24i+48(-1)}{16-64(-1)} && \text{Simplify}\\
&\frac{8-16i-24i-48}{16+64} && \text{Combine like terms}\\
&\frac{-40-40i}{80} && \text{Factor out GCF from numerator}\\
&\frac{40(-1-i)}{80} && \text{Reduce out GCF from numerator}\\
&\frac{\cancel{40}(-1-i)}{\cancel{80}^{\,2}} && \text{Simplify}\\
&\frac{-1-i}{2} && \text{Simplified expression}
\end{aligned}$$

10.8.6 Powers of i

Let's take a look at powers of i:

$$\begin{array}{lll}
i^1 = i & i^5 = i & i^9 = i\\
i^2 = -1 & i^6 = -1 & i^{10} = -1\\
i^3 = -i & i^7 = -i & i^{11} = -i\\
i^4 = 1 & i^8 = 1 & i^{12} = 1
\end{array}$$

Notice, after every fourth power of i, the cycle starts over where every power that is a multiple of four is 1. Hence, for any power of i, we can simplify easily by rewriting the power of i as a product of i that is a multiple of four and i raised to a power of at most 3. Let's look at an example.

Example 10.73

Simplify: i^{35}

Solution.

Notice the power is 35, which equals 32 plus 3. We can rewrite the power as a sum of 32 and 3, then the expression as a product.

i^{35}	Rewrite the power as a sum with the largest multiple of four
i^{32+3}	Rewrite as a product using product rule of exponents
$i^{32} \cdot i^3$	Simplify
$1 \cdot -i$	Multiply
$-i$	Simplified expression

Note

To find where to split the power of i, we could divide the power by four. Then use the remainder to evaluate the expression. For example, in Example 10.73, we could divide 35 by 4:

$$35 \div 4 = 8\,\text{R}3$$

Then use the remainder 3 to evaluate i^{35}, i.e.,

$$i^{35} = i^3 = -i$$

Example 10.74

Simplify: i^{73}

Solution.

Using the note above, let's take the power 73 and divide by 4:

$$73 \div 4 = 18\,\text{R}1$$

We can use the remainder to rewrite i^{73} as

$$i^{73} = i^1 = i$$

Hence $i^{73} = i$.

We can use the remainder method or the method displayed in Example 10.73.

Example 10.75

Simplify: i^{124}

Solution.

Using the remainder method, let's take 124 and divide by 4:

$$124 \div 4 = 31 \text{ R}0$$

We can use the remainder to rewrite i^{124} as

$$i^{124} = i^{0} = 1$$

Hence $i^{124} = 1$. Notice, the power 124 is a multiple of four, and we know that any power of i that is a multiple of four is one from the cycle for powers of i.

10.8.7 Complex Numbers Homework

Simplify.

1. $\sqrt{-64}$

2. $\sqrt{-9}$

3. $\sqrt{-81}$

4. $\sqrt{-121}$

5. $\sqrt{-100}$

6. $\sqrt{-45}$

7. $\sqrt{-90}$

8. $\sqrt{-420}$

9. $\sqrt{-245}$

10. $(6i)(-8i)$

11. $(-5i)(8i)$

12. $(-8i)(4i)$

13. $(-3i)(-4i)$

14. $(-9i)(-4i)$

15. $(3i)(5i)$

16. $(-7i)^2$

17. $\sqrt{-10}\cdot\sqrt{-2}$

18. $\sqrt{-12}\cdot\sqrt{-2}$

19. $-7-\sqrt{-16}$

20. $-3+\sqrt{-121}$

21. $2-\sqrt{-25}$

22. $4+\sqrt{-4}$

23. $\dfrac{3+\sqrt{-27}}{6}$

24. $\dfrac{8-\sqrt{-16}}{4}$

25. $\dfrac{-4-\sqrt{-8}}{-4}$

26. $\dfrac{6+\sqrt{-32}}{4}$

27. $\dfrac{25-\sqrt{-75}}{5}$

28. $\dfrac{-10+\sqrt{-250}}{5}$

29. $\dfrac{15+\sqrt{-108}}{6}$

30. $\dfrac{12+\sqrt{-192}}{8}$

31. $3-(-8+4i)$

32. $(7i)-(3-2i)$

33. $(-6i)-(3+7i)$

34. $(3-3i)+(-7-8i)$

35. $(i)-(2+3i)-6$

36. $(6+5i)^2$

37. $(-7-4i)(-8+6i)$

38. $(-4+5i)(2-7i)$

39. $(-8-6i)(-4+2i)$

40. $(1+5i)(2+i)$

41. $\dfrac{-9+5i}{i}$

42. $\dfrac{-10-9i}{6i}$

43. $\dfrac{-3-6i}{4i}$

44. $\dfrac{10-i}{-i}$

45. $\dfrac{4i}{-10+i}$

46. $\dfrac{8}{7-6i}$

47. $\dfrac{7}{10-7i}$

48. $\dfrac{5i}{-6-i}$

49. $(3i)-(7i)$

50. $5+(-6-6i)$

51. $(-8i)-(7i)-(5-3i)$

52. $(-4-i)+(1-5i)$

53. $(5-4i)+(8-4i)$

54. $(-i)(7i)(4-3i)$

55. $(8i)(-2i)(-2-8i)$

56. $(3i)(-3i)(4-4i)$

57. $-8(4-8i)-2(-2-6i)$

58. $(-6i)(3-2i)-(7i)(4i)$

59. $(-2+i)(3-5i)$

60. $\dfrac{-3+2i}{-3i}$

61. $\dfrac{-4+2i}{3i}$

62. $\dfrac{-5+9i}{9i}$

63. $\dfrac{10}{5i}$

64. $\dfrac{9i}{1-5i}$

65. $\dfrac{4}{4+6i}$

66. $\dfrac{9}{-8-6i}$

67. $\dfrac{8i}{6-7i}$

68. i^{77}

69. i^{48}

70. i^{62}

71. i^{154}

72. i^{251}

73. i^{68}

74. i^{181}

75. i^{51}

10.9 Radicals: Answers to the Homework Exercises

Simplify Radicals

1. $7\sqrt{5}$

3. $2\sqrt{3}$

5. $48\sqrt{2}$

7. $8\sqrt{3n}$

9. $6x\sqrt{7}$

11. $-56x^2$

13. $3xy\sqrt{5}$

15. $8x^2y^2\sqrt{5}$

17. $35xy\sqrt{5y}$

19. $-48x^2z^2y\sqrt{5}$

21. $-12p\sqrt{6mn}$

23. 14

25. $20\sqrt{2}$

27. $-21\sqrt{7}$

29. $10n\sqrt{n}$

31. $-20p^2\sqrt{7}$

33. $32p\sqrt{7}$

35. $16a^2b\sqrt{2}$

37. $56\sqrt{2mn}$

39. $-30y^2x\sqrt{2x}$

41. $-4yz\sqrt{2xz}$

43. $(\sqrt[5]{m})^3$

45. $(\sqrt{7x})^3$

47. $(6x)^{-\frac{3}{2}}$

49. $n^{-\frac{7}{4}}$

51. 4

53. 8

Add and Subtract Radicals

1. $6\sqrt{5}$

3. $-5\sqrt{6}$

5. $-8\sqrt{2}$

7. $-2\sqrt{2}$

9. $-3\sqrt{6}-\sqrt{3}$

11. $-4\sqrt{6}+4\sqrt{5}$

13. $2\sqrt[3]{2}$

15. $\sqrt[4]{2}-3\sqrt[4]{3}$

17. $2\sqrt[4]{2}+\sqrt[4]{3}+6\sqrt[4]{4}$

19. $4\sqrt[5]{5}-4\sqrt[5]{6}$

21. $-3\sqrt{6}-5\sqrt{3}$

23. $-3\sqrt{3}$

25. $-6\sqrt{6}+9\sqrt{3}$

27. $8\sqrt{5}-\sqrt{3}$

29. $3\sqrt{2}+3\sqrt{6}$

31. $-\sqrt{5}-3\sqrt{6}$

33. $6\sqrt[3]{5}-3\sqrt[3]{3}$

35. $5\sqrt[4]{6}+2\sqrt[4]{4}$

37. $-2\sqrt[4]{3}-9\sqrt[4]{5}-3\sqrt[4]{2}$

39. $-11\sqrt[7]{2}-2\sqrt[7]{5}$

Multiply and Divide Radicals

1. $-48\sqrt{5}$

3. $2x^2\sqrt[3]{x}$

5. $-45\sqrt{5}-10\sqrt{15}$

7. $-2-4\sqrt{2}$

9. $6a+a\sqrt{10}+6a\sqrt{6}+2a\sqrt{15}$

11. $\frac{\sqrt{3}}{25}$

13. $\frac{\sqrt{5}}{2}$

15. $\frac{5}{12y^4}$

17. $\frac{\sqrt[3]{10}}{5}$

19. $\frac{5\sqrt[4]{r^2}}{2}$ or $\frac{5\sqrt{r}}{2}$

21. $-25r^2\sqrt{2r}$

23. $5\sqrt{2}+2\sqrt{5}$

25. $5\sqrt{3}-9\sqrt{5v}$

27. $30+8\sqrt{3}+5\sqrt{15}+4\sqrt{5}$

29. $-10\sqrt{m}+25\sqrt{2}+\sqrt{2m}-5$

31. 2

33. $4\sqrt{2}$

35. $\frac{\sqrt{n}}{2}$

37. $\frac{1}{4}$

Rationalize Denominators

1. $\frac{4\sqrt{3}}{9}$
3. 2
5. $\frac{4\sqrt{5}}{5}$
7. $\frac{2+\sqrt{3}}{5}$
9. $\frac{\sqrt{6}-9}{3}$
11. $\frac{10-2\sqrt{2}}{23}$
13. $3-\sqrt{5}$
15. $\sqrt{2}-1$
17. $\sqrt{a}$
19. $4-2\sqrt{3}+2\sqrt{6}-3\sqrt{2}$
21. $3\sqrt{2}+2\sqrt{3}$
23. $\frac{-1+\sqrt{5}}{4}$
25. $\frac{\sqrt{3}-1}{4}$
27. $\frac{\sqrt{30}-2\sqrt{3}}{18}$
29. $\frac{2\sqrt{3}+\sqrt{2}}{2}$
31. $\frac{\sqrt{5}-\sqrt{3}}{2}$
33. $3+2\sqrt{3}$
35. $3-2\sqrt{2}$
37. $\frac{2\sqrt{5}-2\sqrt{15}+\sqrt{3}+3}{-2}$
39. $\frac{a\sqrt{b}+b\sqrt{a}}{a-b}$
41. $\frac{2\sqrt{5}-5\sqrt{2}-10+5\sqrt{10}}{30}$
43. $\frac{8+3\sqrt{6}}{10}$

Radicals with Mixed Indices

1. $\sqrt[4]{4x^2y^3}$
3. $\frac{\sqrt[3]{36xy}}{3y}$
5. $\sqrt[4]{x^3y^2z}$
7. $\sqrt{3xy^3}$
9. $\sqrt[5]{x^3y^4z^2}$
11. $\sqrt{5y}$
13. $\sqrt[6]{5400}$
15. $\sqrt[6]{x^3(x-2)^2}$
17. $\sqrt[12]{x^{11}y^{10}}$
19. $a\sqrt[4]{a}$
21. $xy\sqrt[6]{xy^5}$
23. $x\sqrt[12]{59049xy^{11}z^{10}}$
25. $\sqrt[12]{a^5}$
27. $\sqrt[10]{ab^9c^7}$
29. $\sqrt[15]{(2x+1)^4}$
31. $\sqrt[15]{27y^5z^5}$
33. $\sqrt[10]{4a^9b^9}$
35. $\sqrt[30]{x^{22}y^{11}z^{27}}$
37. $a\sqrt[12]{a^5}$
39. $2xy^2\sqrt[6]{2x^5y}$
41. $4x(y+z)^3\sqrt[6]{2x(y+z)}$
43. $\frac{\sqrt[15]{a^7b^{11}}}{b}$
45. $\sqrt[12]{(2+5x)^5}$

Radical Equations

1. 3
3. ± 2
5. 5
7. 5
9. 3
11. 3
13. 7
15. 21
17. **a)** 79.5 inches; **b)** 145.7 pounds

Solving with Rational Exponents

1. $\pm 5\sqrt{3}$
3. $\pm 2\sqrt{6}$
5. -1
7. -7
9. $-\frac{3}{8}, -\frac{5}{8}$
11. $-\frac{34}{3}, -10$
13. -2
15. $-3, 11$
17. $\frac{-1\pm 3\sqrt{2}}{2}$
19. $-\frac{11}{2}, \frac{5}{2}$
21. $\frac{9}{8}$
23. 3

Complex Numbers

1. $8i$

3. $9i$

5. $10i$

7. $3i\sqrt{10}$

9. $7i\sqrt{5}$

11. 40

13. -12

15. -15

17. $-2\sqrt{5}$

19. $-7-4i$

21. $2-5i$

23. $\frac{1+i\sqrt{3}}{2}$

25. $\frac{2+i\sqrt{2}}{2}$

27. $5-i\sqrt{3}$

29. $\frac{5+2i\sqrt{3}}{2}$

31. $11-4i$

33. $-3-13i$

35. $-8-2i$

37. $80-10i$

39. $44+8i$

41. $9i+5$

43. $\frac{3i-6}{4}$

45. $\frac{-40i+4}{101}$

47. $\frac{70+49i}{149}$

49. $-4i$

51. $5-12i$

53. $13-8i$

55. $-32-128i$

57. $-28+76i$

59. $-1+13i$

61. $\frac{4i+2}{3}$

63. $-2i$

65. $\frac{4-6i}{13}$

67. $\frac{48i-56}{85}$

69. 1

71. -1

73. 1

75. $-i$

Chapter 11

Quadratic Equations and Applications

➥ Chapter Objectives

By the end of this chapter, the student should be able to

- Solve quadratics by the square root property, completing the square, and using the quadratic formula
- Graph a quadratic function by using properties or transformations
- Solve quadratic inequalities by graphing, or algebraically
- Find the extreme value of a quadratic function
- Solve applications and functions using quadratic functions

We might recognize a *quadratic equation* from the factoring chapter as a *trinomial equation.* Although, it may seem that they are the same, they aren't the same. Trinomial equations are equations with any three terms. These terms can be any three terms where the degree of each term can vary. On the other hand, *quadratic equations* are equations with specific degrees on each term.

Definition

A ***quadratic equation*** is a polynomial equation of the form

$$ax^2 + bx + c = 0,$$

where ax^2 is called the *leading term*, bx is called the *linear term*, and c is called the constant coefficient (or constant term). Additionally, $a \neq 0$.

In this chapter, we discuss quadratic equations and its applications. We learn three techniques for solving quadratic equations:

- Square root property
- Completing the square
- Quadratic Formula

Recall, we also have the technique of factoring. After this chapter, we solve quadratic equations by using any of the techniques we have discussed in this textbook. The first technique is using the *Square root property.*

11.1 The Square root property

Let's take a simple quadratic equation, $x^2 = a$ and solve:

$$\begin{aligned} x^2 &= a && \text{Take the square root of both sides} \\ |x| &= \pm\sqrt{a} && \text{Apply absolute value definition} \\ x &= \pm\sqrt{a} && \text{Rewrite as two solutions} \\ x = \sqrt{a} \text{ or } x &= -\sqrt{a} && \text{Solution} \end{aligned}$$

This is the square root property.

Square root property

$$x^2 = a \text{ if and only if } x = \pm\sqrt{a}$$

In other words,

$$x^2 = a \text{ if and only if } x = \sqrt{a} \text{ or } x = -\sqrt{a}$$

Example 11.1

Solve: $x^2 = 81$

Solution.

We could rewrite the equation so that 81 is on the left and then solve by factoring. However, for the sake of the property, we solve this equation by applying the square root property.

$$\begin{aligned} x^2 &= 81 && \text{The } x^2 \text{ is isolated and we apply the square root property} \\ x &= \pm\sqrt{81} && \text{Simplify} \\ x &= \pm 9 && \text{Rewrite as two solutions} \\ x = 9 \text{ or } x &= -9 && \text{Solution} \end{aligned}$$

Notice, we could write the solution two ways: ± 9, or, alternatively, 9 or -9. As the problems become more challenging, it is common practice to write the solutions as two solutions.

Example 11.2

Solve: $x^2 = 44$

Solution.

Notice, even if we moved 44 to the left and tried to factor, we couldn't because 44 is not a perfect square. Hence, we need the square root property to solve.

$$\begin{aligned} x^2 &= 44 && \text{The } x^2 \text{ is isolated and we apply the square root property} \\ x &= \pm\sqrt{44} && \text{Simplify} \\ x &= \pm\sqrt{4 \cdot 11} && \text{Apply the product property} \\ x &= \pm 2\sqrt{11} && \text{Rewrite as two solutions} \\ x = 2\sqrt{11} \text{ or } x &= -2\sqrt{11} && \text{Solution} \end{aligned}$$

World Note

In 1545, French mathematician, *Gerolamo Cardano*, published his book *The Great Art, or the Rules of Algebra*. It included the solution of an equation with a fourth power, but it was considered absurd by many to take a quantity to the fourth power because there are only three dimensions.

Example 11.3

Solve: $(x+4)^2 = 25$

Solution.

Even though the base has changed from x to $(x+4)$, the method doesn't change. Hence, we will apply the square root property to solve as long as the base is isolated.

$$\begin{aligned} (x+4)^2 &= 25 && \text{The } (x+4)^2 \text{ is isolated and we apply the square root property} \\ x+4 &= \pm\sqrt{25} && \text{Isolate } x \\ x &= -4 \pm \sqrt{25} && \text{Simplify } \sqrt{25} \\ x &= -4 \pm 5 && \text{Rewrite as two solutions} \\ x = -4+5 \text{ or } x &= -4-5 && \text{Evaluate} \\ x = -1 \text{ or } x &= -9 && \text{Solution} \end{aligned}$$

Here, we rewrote the solution as two different solutions in order to solve.

Example 11.4

Solve: $(6x-9)^2 = 45$

Solution.

Even though the base has changed from x to $(6x-9)$, the method doesn't change. Hence, we will apply the square root property to solve as long as the base is isolated.

$$\begin{aligned} (6x-9)^2 &= 45 && \text{The } (6x-9)^2 \text{ is isolated and we apply the square root property} \\ 6x-9 &= \pm\sqrt{45} && \text{Isolate the variable term} \\ 6x &= 9 \pm \sqrt{45} && \text{Simplify } \sqrt{45} \\ 6x &= 9 \pm \sqrt{9\cdot 5} && \text{Apply the product property} \\ 6x &= 9 \pm 3\sqrt{5} && \text{Solve for } x \\ x &= \frac{9 \pm 3\sqrt{5}}{6} && \text{Factor a GCF} \\ x &= \frac{\cancel{3}(3 \pm \sqrt{5})}{\cancelto{2}{6}} && \text{Simplify} \\ x &= \frac{3 \pm \sqrt{5}}{2} && \text{Solution} \end{aligned}$$

Here, we leave the solution with the $\pm$ since the radicand was not a perfect square. Usually, when the radical is completely reduced out of the equation, we separate the solutions. Otherwise, we leave it as is.

11.1.1 Isolate the squared term

Let's take a look at examples where the leading term, or squared term, is not isolated. Recall, the squared term must be isolated in order to apply the square root property.

Example 11.5

Solve: $5(3x-6)^2+7=27$

Solution.

We first need to isolate $(3x-6)^2$ in order to apply the square root property. Then we can solve as usual.

$$\begin{aligned}
5(3x-6)^2+7&=27 && \text{Isolate the variable term}\\
5(3x-6)^2&=20 && \text{Isolate } (3x-6)^2\\
(3x-6)^2&=4 && \text{Apply the square root property}\\
3x-6&=\pm\sqrt{4} && \text{Isolate the variable term}\\
3x&=6\pm\sqrt{4} && \text{Simplify } \sqrt{4}\\
3x&=6\pm 2 && \text{Solve for } x\\
x&=\frac{6\pm 2}{3} && \text{Rewrite as two solutions}\\
x=\frac{6+2}{3} \text{ or } x&=\frac{6-2}{3} && \text{Evaluate}\\
x=\frac{8}{3} \text{ or } x&=\frac{4}{3} && \text{Solution}
\end{aligned}$$

Notice the radicand was a perfect square and so we were able to write the solutions as two separate numbers.

Example 11.6

Solve: $5(r+4)^2+1=626$

Solution.

We first need to isolate $(r+4)^2$ in order to apply the square root property. Then we can solve as usual.

$$\begin{aligned}
5(r+4)^2+1&=626 && \text{Isolate the variable term}\\
5(r+4)^2&=625 && \text{Isolate } (r+4)^2\\
(r+4)^2&=125 && \text{Apply the square root property}\\
r+4&=\pm\sqrt{125} && \text{Solve for } r\\
r&=-4\pm\sqrt{125} && \text{Simplify } \sqrt{125}\\
r&=-4\pm\sqrt{25\cdot 5} && \text{Apply the product property}\\
r&=-4\pm 5\sqrt{5} && \text{Solution}
\end{aligned}$$

Example 11.7

Solve: $2n^2 + 5 = 4$

Solution.

We first need to isolate n^2 in order to apply the square root property. Then we can solve as usual.

$$\begin{aligned} 2n^2 + 5 &= 4 && \text{Isolate the variable term} \\ 2n^2 &= -4 && \text{Isolate } n^2 \\ n^2 &= -2 && \text{Apply the square root property} \\ n &= \pm\sqrt{-2} && \text{Reduce out an } i \\ n &= \pm i\sqrt{2} && \text{Solution} \end{aligned}$$

Recall, a radicand of a square root that is less than zero is the imaginary part of a complex number. Now that we just discussed complex numbers in the previous chapter, we can solve any type of quadratic equation with real and complex solutions.

11.1.2 Square Root Property Homework

Solve by applying the square root property.

1. $(x-3)^2 = 16$

2. $(x-2)^2 = 49$

3. $(x-7)^2 = 4$

4. $(s-5)^2 = 16$

5. $(p+5)^2 = 81$

6. $(s+3)^2 = 4$

7. $(t+9)^2 = 37$

8. $(a+5)^2 = 87$

9. $(v-2)^2 = 70$

10. $(n-9)^2 = 63$

11. $(v+4)^2 = 63$

12. $(r+1)^2 = 125$

13. $(9r+1)^2 = 9$

14. $(7m-8)^2 = 36$

15. $(3s-6)^2 = 25$

16. $5(k-7)^2 - 6 = 369$

17. $5(z+6)^2 - 10 = 365$

18. $5(g-5)^2 + 13 = 103$

19. $(2s+1)^2 = 0$

20. $(z-4)^2 = 25$

21. $(w+3)^2 = 49$

22. $2n^2 + 7 = 5$

23. $3n^2 + 2n = 2n + 24$

24. $8n^2 - 29 = 25 + 2n^2$

25. $2(r+9)^2 - 19 = 37$

26. $3(n-3)^2 + 2 = 164$

27. $3(y+8)^2 + 12 = 147$

28. $6(4x-4)^2 - 5 = 145$

29. $3(4x+6)^2 - 5 = 103$

30. $7(2x+6)^2 - 5 = 170$

31. $4(7x+6)^2 - 3 = 61$

32. $3(7x+3)^2 + 7 = 55$

33. $5(4x-5)^2 - 2 = 18$

11.2 Completing the square

When solving quadratic equations previously (then known as trinomial equations), we factored to solve. However, recall, not all equations are factorable. Consider the equation $x^2 - 2x - 7 = 0$. This equation isn't factorable, but there are two solutions to this equation: $1 + 2\sqrt{2}$ and $1 - 2\sqrt{2}$. Looking at the form of these solutions, we obtained these types of solutions in the previous section while using the square root property. If we can obtain a perfect square, then we can apply the square root property and solve as usual. This method we use to obtain a perfect square is called *completing the square.*

Recall. Special product formulas for perfect square trinomials:

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \text{or} \quad (a-b)^2 = a^2 - 2ab + b^2$$

We use these formulas to help us solve by completing the square.

11.2.1 Complete the square

We first begin with completing the square and rewriting the trinomial in factored form using the perfect square trinomial formulas.

Example 11.8

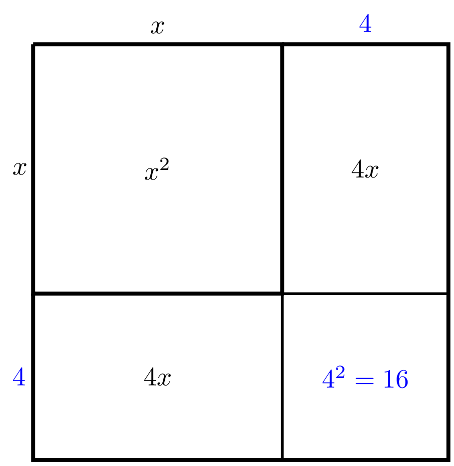
Complete the square by finding c: $x^2 + 8x + c$

Solution.

There are a couple of ways to complete the square. The first way is to mentally think about a number for c such that we can factor the trinomial as a perfect square trinomial, i.e.,

$$x^2 + 8x + c = (x + \underline{\quad\quad})^2$$

Some might see that this number $c = 16$ if they are keen at factoring. Notice, $(x+4)^2 = x^2 + 8x + 16$. Another way is to, literally, complete the square:



Notice the square has all components of the perfect square trinomial. Hence, we can see the dimensions of this square to be

$$(x+4) \times (x+4)$$

which is

$$(x+4)^2$$

and the missing constant coefficient is 16, the square of 4. Using one of these methods will suffice depending on the type of learner. Some students enjoy the geometric relationship between the quadratic equation and the square, and some enjoy the algebraic method. It is up to the discretion of the student.

Thus, we see $c = 16$ and the perfect square trinomial $x^2 + 8x + 16$ is factored into $(x + 4)^2$.

In example 11.8, notice c was derived from the middle term $8x$. Looking at the square, we see the linear term's coefficient 8 was divided in half and squared (because we were finding the area of the bottom right square). This is exactly the method we use for all problems when completing the square.

Note

To complete the square of any trinomial, we always square half of the linear term's coefficient, i.e.,

$$\left(\frac{b}{2}\right)^2 \quad \text{or} \quad \left(\frac{1}{2}b\right)^2$$

We usually use the second expression when the middle term's coefficient is a fraction.

Example 11.9

Complete the square by finding c: $x^2 - 7x + c$

Solution.

To obtain c, we use the formula above $\left(\frac{b}{2}\right)^2$.

$$\begin{aligned} x^2 - 7x + c & \qquad b = -7;\ \text{apply formula} \left(\frac{b}{2}\right)^2 \\ x^2 - 7x + \left(\frac{-7}{2}\right)^2 & \qquad \text{Simplify } c \\ x^2 - 7x + \frac{49}{4} & \qquad \text{Perfect square trinomial} \end{aligned}$$

Thus, $c = \frac{49}{4}$. Rewriting this perfect square trinomial in factored form, we get

$$x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$$

Example 11.10

Complete the square by finding c: $x^2 + \frac{5}{3}x + c$

Solution.

To obtain c, we use the formula above $\left(\frac{1}{2}b\right)^2$ since the linear term's coefficient is a fraction.

$$\begin{aligned} x^2+\frac{5}{3}x+c &\qquad b=\frac{5}{3};\text{ apply formula }\left(\frac{1}{2}b\right)^2 \\ x^2+\frac{5}{3}x+\left(\frac{1}{2}\cdot\frac{5}{3}\right)^2 &\qquad \text{Simplify } c \\ x^2+\frac{5}{3}x+\frac{25}{36} &\qquad \text{Perfect square trinomial} \end{aligned}$$

Thus, $c=\frac{25}{36}$. Rewriting this perfect square trinomial in factored form, we get

$$x^2+\frac{5}{3}x+\frac{25}{36}=\left(x+\frac{5}{6}\right)^2$$

World Note

The Chinese in 200 BC were the first known culture group to use a method similar to completing the square, but their method was only used to calculate positive roots.

Steps to solving quadratic equations by completing the square

Given a quadratic equation $x^2+bx+c=0$, we can use the following method to solve for x.

Step 1. Rewrite the quadratic equation so that the coefficient of the leading term is one, and the original constant coefficient is on the opposite side of the equal sign from the leading and linear terms.

Step 2. Complete the square, i.e., $\left(\frac{b}{2}\right)^2$ or $\left(\frac{1}{2}b\right)^2$, and add the result to both sides of the quadratic equation.

Step 3. Rewrite the perfect square trinomial in factored form.

Step 4. Solve using the square root property.

Step 5. Verify the solution(s).

11.2.2 Solve quadratic equations by completing the square, $a=1$

Recall. When the leading coefficient is one, i.e., $a=1$, of the quadratic equation $ax^2+bx+c=0$, we have the quadratic equation $x^2+bx+c=0$.

Example 11.11

Solve: $x^2+10x=-24$

Solution.

Step 1. Rewrite the quadratic equation so that the coefficient of the leading term is one, and the original constant coefficient is on the opposite side of the equal sign from the leading and linear terms. Notice, the first step is done for us:

$$x^2 + 10x = -24$$

Step 2. Complete the square, i.e., $\left(\frac{b}{2}\right)^2$ or $\left(\frac{1}{2}b\right)^2$, and add the result to both sides of the quadratic equation.

$$\begin{aligned} x^2 + 10x &= -24 && b = 10;\ \text{apply formula } \left(\frac{b}{2}\right)^2 \\ x^2 + 10x + \left(\frac{10}{2}\right)^2 &= -24 + \left(\frac{10}{2}\right)^2 && \text{Simplify} \\ x^2 + 10x + 25 &= -24 + 25 && \text{Perfect square trinomial} \end{aligned}$$

Step 3. Rewrite the perfect square trinomial in factored form.

$$\begin{aligned} x^2 + 10x + 25 &= -24 + 25 && \text{Perfect square trinomial} \\ (x+5)^2 &= 1 && \text{Factored form} \end{aligned}$$

Step 4. Solve using the square root property.

$$\begin{aligned} (x+5)^2 &= 1 && \text{Apply the square root property} \\ x + 5 &= \pm\sqrt{1} && \text{Isolate the variable} \\ x &= -5 \pm \sqrt{1} && \text{Rewrite as two solutions} \\ x = -5 + 1 \text{ or } x &= -5 - 1 && \text{Evaluate} \\ x = -4 \text{ or } x &= -6 && \text{Solution} \end{aligned}$$

Step 5. Verify the solution(s).

$$\begin{array}{rr} x^2 + 10x = -24 & x^2 + 10x = -24 \\ (-4)^2 + 10(-4) \stackrel{?}{=} -24 & (-6)^2 + 10(-6) \stackrel{?}{=} -24 \\ 16 - 40 \stackrel{?}{=} -24 & 36 - 60 \stackrel{?}{=} -24 \\ -24 = -24\ \checkmark & -24 = -24\ \checkmark \end{array}$$

Thus, $x = -4$ and $x = -6$ are the solutions.

Example 11.12

Solve: $n^2 + 8n + 4 = 0$

Solution.

Step 1. Rewrite the quadratic equation so that the coefficient of the leading term is one, and the original constant coefficient is on the opposite side of the equal sign from the leading and linear terms.

$$\begin{aligned} n^2 + 8n + 4 &= 0 \\ n^2 + 8n &= -4 \end{aligned}$$

Step 2. Complete the square, i.e., $\left(\frac{b}{2}\right)^2$ or $\left(\frac{1}{2}b\right)^2$, and add the result to both sides of the quadratic equation.

$$\begin{aligned} n^2 - 8n &= -4 && b = -8;\ \text{apply formula} \left(\frac{b}{2}\right)^2 \\ n^2 - 8n + \left(\frac{-8}{2}\right)^2 &= -4 + \left(\frac{-8}{2}\right)^2 && \text{Simplify} \\ n^2 - 8n + 16 &= -4 + 16 && \text{Perfect square trinomial} \end{aligned}$$

Step 3. Rewrite the perfect square trinomial in factored form.

$$\begin{aligned} n^2 - 8n + 16 &= -4 + 16 && \text{Perfect square trinomial} \\ (n-4)^2 &= 12 && \text{Factored form} \end{aligned}$$

Step 4. Solve using the square root property.

$$\begin{aligned} (n-4)^2 &= 12 && \text{Apply the square root property} \\ n - 4 &= \pm\sqrt{12} && \text{Isolate the variable} \\ n &= 4 \pm \sqrt{12} && \text{Simplify } \sqrt{12} \\ n &= 4 \pm \sqrt{4 \cdot 3} && \text{Apply product property for radicals} \\ n &= 4 \pm 2\sqrt{3} && \text{Solution} \end{aligned}$$

Step 5. Verify the solution(s). *We leave this step to the student.*

Thus, $n = 4 + 2\sqrt{3}$ and $n = 4 - 2\sqrt{3}$ are the solutions.

11.2.3 Solve quadratic equations by completing the square, $a \neq 1$

Example 11.13

Solve: $3x^2 - 36x + 60 = 0$

Solution.

Step 1. Rewrite the quadratic equation so that the coefficient of the leading term is one, and the original constant coefficient is on the opposite side of the equal sign from the leading and linear terms.

$$\begin{aligned} 3x^2 - 36x + 60 &= 0 \\ 3x^2 - 36x &= -60 \\ 3\left(x^2 - 12x\right) &= 3 \cdot -20 \\ \cancel{3}\left(x^2 - 12x\right) &= \cancel{3} \cdot -20 \\ x^2 - 12x &= -20 \end{aligned}$$

Step 2. Complete the square, i.e., $\left(\frac{b}{2}\right)^2$ or $\left(\frac{1}{2}b\right)^2$, and add the result to both sides of the quadratic equation.

$$x^2 - 12x = -20 \qquad b = -12;\text{ apply formula } \left(\frac{b}{2}\right)^2$$

$$x^2 - 12x + \left(\frac{-12}{2}\right)^2 = -20 + \left(\frac{-12}{2}\right)^2 \qquad \text{Simplify}$$

$$x^2 - 12x + 36 = -20 + 36 \qquad \text{Perfect square trinomial}$$

Step 3. Rewrite the perfect square trinomial in factored form.

$$\begin{aligned} x^2 - 12x + 36 &= -20 + 36 && \text{Perfect square trinomial} \\ (x-6)^2 &= 16 && \text{Factored form} \end{aligned}$$

Step 4. Solve using the square root property.

$$\begin{aligned} (x-6)^2 &= 16 && \text{Apply the square root property} \\ x - 6 &= \pm\sqrt{16} && \text{Isolate the variable} \\ x &= 6 \pm \sqrt{16} && \text{Rewrite as two solutions} \\ x = 6 + 4 \text{ or } x &= 6 - 4 && \text{Evaluate} \\ x = 10 \text{ or } x &= 2 && \text{Solution} \end{aligned}$$

Step 5. Verify the solution(s). *We leave this step to the student.*

Thus, $x = 10$ and $x = 2$ are the solutions.

Example 11.14

Solve: $2k^2 + k - 2 = 0$

Solution.

Step 1. Rewrite the quadratic equation so that the coefficient of the leading term is one, and the original constant coefficient is on the opposite side of the equal sign from the leading and linear terms.

$$\begin{aligned} 2k^2 + k - 2 &= 0 \\ 2k^2 + k &= 2 \\ 2\left(k^2 + \frac{1}{2}k\right) &= 2 \cdot 1 \\ \cancel{2}\left(k^2 + \frac{1}{2}k\right) &= \cancel{2} \cdot 1 \\ k^2 + \frac{1}{2}k &= 1 \end{aligned}$$

Step 2. Complete the square, i.e., $\left(\frac{b}{2}\right)^2$ or $\left(\frac{1}{2}b\right)^2$, and add the result to both sides of the quadratic equation.

$$
\begin{aligned}
k^2+\frac{1}{2}k &= 1 && b=\frac{1}{2};\ \text{apply formula } \left(\frac{1}{2}b\right)^2 \\
k^2+\frac{1}{2}k+\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2 &= 1+\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2 && \text{Simplify} \\
k^2+\frac{1}{2}k+\frac{1}{16} &= 1+\frac{1}{16} && \text{Perfect square trinomial}
\end{aligned}
$$

Step 3. Rewrite the perfect square trinomial in factored form.

$$
\begin{aligned}
k^2+\frac{1}{2}k+\frac{1}{16} &= 1+\frac{1}{16} && \text{Perfect square trinomial} \\
\left(k+\frac{1}{4}\right)^2 &= \frac{17}{16} && \text{Factored form}
\end{aligned}
$$

Step 4. Solve using the square root property.

$$
\begin{aligned}
\left(k+\frac{1}{4}\right)^2 &= \frac{17}{16} && \text{Apply the square root property} \\
k+\frac{1}{4} &= \pm\sqrt{\frac{17}{16}} && \text{Isolate the variable} \\
k &= -\frac{1}{4}\pm\sqrt{\frac{17}{16}} && \text{Simplify } \sqrt{\frac{17}{16}} \\
k &= -\frac{1}{4}\pm\frac{\sqrt{17}}{4} && \text{Same denominator, combine fractions} \\
k &= \frac{-1\pm\sqrt{17}}{4} && \text{Solution}
\end{aligned}
$$

Step 5. Verify the solution(s). *We leave this step to the student.*

Thus, $\dfrac{-1+\sqrt{17}}{4}$ and $\dfrac{-1-\sqrt{17}}{4}$ are the solutions.

11.2.4 Completing the Square Homework

Complete the square and rewrite in factored form.

1. $x^2 - 30x + \_\_\_\_$ **2.** $m^2 - 36m + \_\_\_\_$ **3.** $x^2 - 15x + \_\_\_\_$ **4.** $y^2 - y + \_\_\_\_$

5. $a^2 - 24a + \_\_\_\_$ **6.** $x^2 - 34x + \_\_\_\_$ **7.** $r^2 - \frac{1}{9}r + \_\_\_\_$ **8.** $p^2 - 17p + \_\_\_\_$

Solve each equation by completing the square.

9. $x^2 - 16x + 55 = 0$ **10.** $v^2 - 8v + 45 = 0$ **11.** $6x^2 + 12x + 63 = 0$

12. $5k^2 - 10k + 48 = 0$ **13.** $x^2 + 10x - 57 = 4$ **14.** $n^2 - 16n + 67 = 4$

15. $2x^2 + 4x + 38 = -6$ **16.** $8b^2 + 16b - 37 = 5$ **17.** $x^2 = -10x - 29$

18. $n^2 = -21 + 10n$ **19.** $3k^2 + 9 = 6k$ **20.** $2x^2 + 63 = 8x$

21. $p^2 - 8p = -55$ **22.** $7n^2 - n + 7 = 7n + 6n^2$ **23.** $13b^2 + 15b + 44 = -5 + 7b^2 + 3b$

24. $5x^2 + 5x = -31 - 5x$ **25.** $v^2 + 5v + 28 = 0$ **26.** $7x^2 - 6x + 40 = 0$

27. $k^2 - 7k + 50 = 3$ **28.** $5x^2 + 8x - 40 = 8$ **29.** $m^2 = -15 + 9m$

30. $8r^2 + 10r = -55$ **31.** $-2x^2 + 3x - 5 = -4x^2$ **32.** $5n^2 - 8n + 60 = -3n + 6 + 4n^2$

33. $n^2 - 8n - 12 = 0$ **34.** $b^2 + 2b + 43 = 0$ **35.** $3x^2 - 6x + 47 = 0$

36. $8a^2 + 16a - 1 = 0$ **37.** $p^2 - 16p - 52 = 0$ **38.** $m^2 - 8m - 3 = 6$

39. $6r^2 + 12r - 24 = -6$ **40.** $6n^2 - 12n - 14 = 4$ **41.** $v^2 = 14v + 36$

42. $a^2 - 56 = -10a$ **43.** $5x^2 = -26 + 10x$ **44.** $5n^2 = -10n + 15$

45. $x^2 + 8x + 15 = 8$ **46.** $n^2 + 4n = 12$ **47.** $-3r^2 + 12r + 49 = -6r^2$

48. $8n^2 + 16n = 64$ **49.** $b^2 + 7b - 33 = 0$ **50.** $4x^2 + 4x + 25 = 0$

51. $a^2 - 5a + 25 = 3$ **52.** $2p^2 - p + 56 = -8$ **53.** $n^2 - n = -41$

54. $3x^2 - 11x = -18$ **55.** $4b^2 - 15b + 56 = 3b^2$ **56.** $10v^2 - 15v = 27 + 4v^2 - 6v$

11.3 Quadratic formula

The *quadratic formula* is derived from the method of completing the square. If we took a general quadratic equation

$$ax^2 + bx + c = 0$$

and solved for x by completing the square, we would obtain the *quadratic formula.* Let's try this.

Example 11.15

Solve for x by completing the square: $ax^2 + bx + c = 0$

Solution.

First, we should rewrite the equation so that the leading coefficient is one and c is on the other side.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ ax^2 + bx &= -c \\ a\left(x^2 + \frac{b}{a}\right) &= -c \\ x^2 + \frac{b}{a}x &= \frac{-c}{a} \end{aligned}$$

Next, we take $\frac{b}{a}$ and form $\left(\frac{b}{2a}\right)^2$.

$$\begin{aligned} x^2 + \frac{b}{a}x + \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 &= \frac{-c}{a} + \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 \\ x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{-c}{a} + \frac{b^2}{4a^2} \end{aligned}$$

Let's factor the left side and combine fractions on the right:

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{-c}{a}\cdot\frac{4a}{4a} + \frac{b^2}{4a^2} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{-4ac}{4a^2} + \frac{b^2}{4a^2} \end{aligned}$$

We can apply the square root property and solve as usual:

$$\begin{aligned} \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} && \text{Apply the square root property} \\ x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} && \text{Simplify} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \text{Isolate } x \end{aligned}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \qquad \text{Same denominator; combine fractions}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \text{Solution}$$

Quadratic formula

Let a, b be coefficients of x^2, x, respectively, and c be the constant coefficient of the quadratic equation $ax^2 + bx + c = 0$. Then

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

is called the ***quadratic formula***. The *quadratic formula* is a formula for solving quadratic equations in terms of the coefficients.

World Note

Indian mathematician, *Brahmagupta*, gave the first explicit formula for solving quadratics in 628 AD. However, at that time, mathematics was not written with variables and symbols, so the formula he gave was, "To the absolute number multiplied by four times the square, add the square of the middle term; the square root of the same, less the middle term, being divided by twice the square is the value." Mathematically, this would translate to

$$\frac{\sqrt{4ac+b^2}-b}{2a}$$

as the solution to the equation $ax^2 + bx = c$.

11.3.1 Apply the quadratic formula

Example 11.16

Solve: $x^2 + 3x + 2 = 0$

Solution.

We may note that we can solve this equation by factoring. However, we will use the quadratic formula and later compare.

$$x^2 + 3x + 2 = 0 \qquad \text{Apply quadratic formula}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \text{Plug-n-chug } a = 1, b = 3, c = 2$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} \qquad \text{Simplify using order of operations}$$

$$x = \frac{-3 \pm \sqrt{9-8}}{2}$$

$$x = \frac{-3 \pm \sqrt{1}}{2}$$

$$x = \frac{-3 \pm 1}{2} \quad \text{Rewrite as two solutions}$$

$$x = \frac{-3+1}{2} \text{ or } \frac{-3-1}{2} \quad \text{Evaluate}$$

$$x = -1 \text{ or } x = -2 \quad \text{Solution}$$

Let's compare with factoring the equation:

$$\begin{aligned} x^2 + 3x + 2 &= 0 \\ (x+1)(x+2) &= 0 \\ x + 1 = 0 &\text{ or } x + 2 = 0 \\ x = -1 &\text{ or } x = -2 \end{aligned}$$

Notice, factoring would have been much quicker than using the quadratic formula.

Note

When we can factor the quadratic equation, we should, and when the equation isn't factorable, we should use quadratic formula.

11.3.2 Make equal to zero

Example 11.17

Solve: $25x^2 = 30x + 11$

Solution.

We first rewrite the equation so that zero is on one side of the equation. Then we can solve as usual.

$$\begin{aligned} 25x^2 &= 30x + 11 && \text{Rewrite where zero is on one side} \\ 25x^2 - 30x - 11 &= 0 && \text{Apply the quadratic formula} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Plug-n-chug } a = 25, b = -30, c = -11 \\ x &= \frac{-(-30) \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)} && \text{Simplify using order of operations} \\ x &= \frac{30 \pm \sqrt{900 + 1100}}{50} && \\ x &= \frac{30 \pm \sqrt{2000}}{50} && \text{Rewrite } \sqrt{2000} \\ x &= \frac{30 \pm \sqrt{400 \cdot 5}}{50} && \text{Apply product property of radicals} \\ x &= \frac{30 \pm 20\sqrt{5}}{50} && \text{Factor the numerator} \end{aligned}$$

$$x = \frac{\cancel{10}(3 \pm 2\sqrt{5})}{\cancel{50}^{5}} \quad \text{Reduce by a factor of 10}$$

$$x = \frac{3 \pm 2\sqrt{5}}{5} \quad \text{Solution}$$

We can see that the equation wasn't factorable, so we applied the quadratic formula.

Example 11.18

Solve: $3x^2 + 4x + 8 = 2x^2 + 6x - 5$

Solution.

We first rewrite the equation so that zero is on one side of the equation. Then we can solve as usual.

$$\begin{aligned}
3x^2 + 4x + 8 &= 2x^2 + 6x - 5 && \text{Rewrite where zero is on one side}\\
x^2 - 2x + 13 &= 0 && \text{Apply the quadratic formula}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Plug-n-chug } a = 1, b = -2, c = 13\\
x &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(13)}}{2(1)} && \text{Simplify using order of operations}\\
x &= \frac{2 \pm \sqrt{4 - 52}}{2} &&\\
x &= \frac{2 \pm \sqrt{-48}}{2} && \text{Rewrite the radical using } i\\
x &= \frac{2 \pm i\sqrt{48}}{2} && \text{Simplify the radical}\\
x &= \frac{2 \pm 4i\sqrt{3}}{2} && \text{Factor a 2 from the numerator}\\
x &= \frac{2(1 \pm 2i\sqrt{3})}{2} && \text{Reduce by a factor of 2}\\
x &= \frac{\cancel{2}(1 \pm 2i\sqrt{3})}{\cancel{2}} && \text{Rewrite}\\
x &= 1 \pm 2i\sqrt{3} && \text{Solutions}
\end{aligned}$$

When there is a negative value as the radicand, we rewrite the radical using the imaginary unit and the solutions are non-real numbers.

11.3.3 When the linear term is zero

Example 11.19

Solve: $3x^2 - 7 = 0$

Solution.

If the term is missing from the quadratic equation, we solve the equation by using the quadratic formula and plug-n-chug zero for that term. If the linear term is missing, then $b = 0$, and if the constant term is missing, then $c = 0$.

$$\begin{aligned}
3x^2 - 7 &= 0 && \text{Apply the quadratic formula}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Plug-n-chug } a = 3, b = 0, c = -7\\
x &= \frac{-0 \pm \sqrt{0^2 - 4(3)(-7)}}{2(3)} && \text{Simplify using order of operations}\\
x &= \frac{\pm\sqrt{84}}{6} && \text{Rewrite } \sqrt{84}\\
x &= \frac{\pm\sqrt{4 \cdot 21}}{6} && \text{Apply product property of radicals}\\
x &= \frac{\pm\cancel{2}\sqrt{21}}{\cancelto{3}{6}} && \text{Reduce by a factor of 2}\\
x &= \frac{\pm\sqrt{21}}{3} && \text{Solution}
\end{aligned}$$

Note

When we can factor the quadratic equation, we should, and when the equation isn't factorable, we should use the quadratic formula.

If the *linear term* is missing, then $b = 0$, and we can solve by the square root property. If the *constant term* is missing, then $c = 0$, then we can solve by factoring.

11.3.4 Quadratic Formula Homework

Solve each equation by applying the quadratic formula.

1. $4a^2 - 6 = 0$

2. $2x^2 - 8x - 2 = 0$

3. $2m^2 - 3 = 0$

4. $3r^2 - 2r - 1 = 0$

5. $4n^2 - 36 = 0$

6. $v^2 - 4v - 5 = -8$

7. $2a^2 + 3a + 14 = 6$

8. $3k^2 + 3k - 4 = 7$

9. $7x^2 + 3x - 16 = -2$

10. $2p^2 + 6p - 16 = 4$

11. $3n^2 + 3n = -3$

12. $2x^2 = -7x + 49$

13. $5x^2 = 7x + 7$

14. $8n^2 = -3n - 8$

15. $2x^2 + 5x = -3$

16. $4a^2 - 64 = 0$

17. $4p^2 + 5p - 36 = 3p^2$

18. $-5n^2 - 3n - 52 = 2 - 7n^2$

19. $7r^2 - 12 = -3r$

20. $2n^2 - 9 = 4$

21. $3k^2 + 2 = 0$

22. $6n^2 - 1 = 0$

23. $5p^2 + 2p + 6 = 0$

24. $2x^2 - 2x - 15 = 0$

25. $3b^2 + 6 = 0$

26. $2x^2 + 4x + 12 = 8$

27. $6n^2 - 3n + 3 = -4$

28. $4x^2 - 14 = -2$

29. $4n^2 + 5n = 7$

30. $m^2 + 4m - 48 = -3$

31. $3b^2 - 3 = 8b$

32. $3r^2 + 4 = -6r$

33. $6a^2 = -5a + 13$

34. $6v^2 = 4 + 6v$

35. $x^2 = 8$

36. $2k^2 + 6k - 16 = 2k$

37. $12x^2 + x + 7 = 5x^2 + 5x$

38. $7m^2 - 6m + 6 = -m$

39. $3x^2 - 3 = x^2$

40. $6b^2 = b^2 + 7 - b$

11.4 Graph quadratic functions

Let's recall the ***parabola*** from the *Functions* chapter.

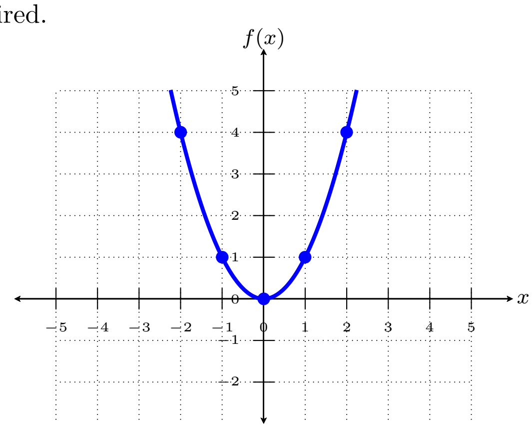
Example 11.20

Graph $f(x) = x^2$.

Solution.

Let's pick five x-coordinates, and find corresponding y-values. Each x-value being positive, negative, and zero. This is common practice, but not required.

x	$f(x) = x^2$	$(x, f(x))$
-2	$f(-2) = 4$	$(-2, 4)$
-1	$f(-1) = 1$	$(-1, 1)$
0	$f(0) = 0$	$(0, 0)$
1	$f(1) = 1$	$(1, 1)$
2	$f(2) = 4$	$(2, 4)$



Plot the five ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. This graph is called a *parabola* and since this function is quite common for the x^2-form, we call it a ***quadratic (square) function***.

Since quadratic functions have a leading term that contains x^2, then a quadratic function's graph is called a ***parabola*** just like in the *Functions* chapter.

Definition

A ***quadratic function*** is a polynomial function of the form

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$.

In example 11.20, we plotted points and connected the dots. This is one way of graphing quadratic functions, but not the most efficient. Hence, we can easily graph quadratic functions by finding key elements of the function: vertex, x-intercepts, and the y-intercept.

11.4.1 Vertex of a quadratic function

Definition

The ***vertex*** of a quadratic function $f(x) = ax^2 + bx + x$ is given by

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

Example 11.21

Find the vertex of $f(x) = x^2 - 3x - 4$.

Solution.

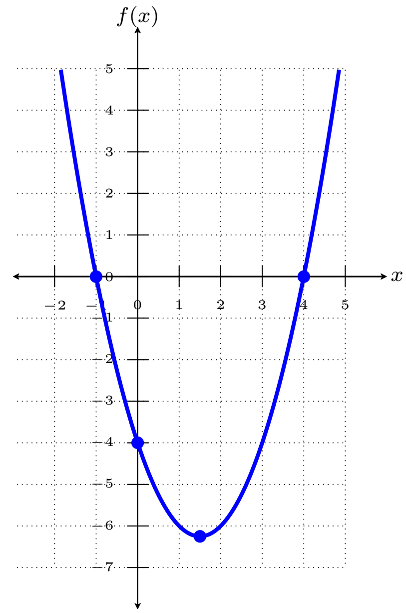
The x-coordinate of the vertex is $-\dfrac{b}{2a}$ given by the definition. In this case, $a = 1$, $b = -3$, and $c = -4$. Hence,

$$x = -\frac{b}{2a} = -\frac{-3}{2(1)} = \frac{3}{2}$$

The y-coordinate of the vertex is

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 4 = -\frac{25}{4}$$

Thus, the vertex of $f(x)$ is at $\left(\dfrac{3}{2}, -\dfrac{25}{4}\right)$. Let's take a look at the graph and verify this is the location of the vertex. We see the vertex is, in fact, located at $\left(\dfrac{3}{2}, -\dfrac{25}{4}\right)$. Additionally, we see the parabola intersects the x-axis at $x = -1$ and $x = 4$, and the y-axis at $(0, -4)$.



11.4.2 Graph quadratic functions by its properties

Properties of a quadratic function

To graph a quadratic function, $f(x) = ax^2 + bx + x$, by its properties, we obtain key properties.

Property 1. The direction of the parabola.

- If $a > 0$, then the graph is an upward parabola.
- If $a < 0$, then the graph is a downward parabola.

Property 2. The vertex: $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$

Property 3. The y-intercept: $(0, f(0))$

Property 4. The x-intercepts: $(x, 0)$, i.e., where $f(x) = 0$, also known as the zeros of $f(x)$.

Property 5. The axis of symmetry: $x = -\dfrac{b}{2a}$

Note

The ***axis of symmetry*** is a vertical line that intersects the vertex of the parabola. Hence, the line $x = -\dfrac{b}{2a}$. The axis of symmetry essentially "cuts" the parabola in half and the parabola is symmetrical about this axis.

Example 11.22

Using the properties, graph $f(x) = x^2 + 4x + 3$.

Solution.

Property 1. *The direction of the parabola.* Since $a = 1$ and $a > 0$, then $f(x)$ is an upwards parabola.

Property 2. *Find the vertex.* We use the formula to find the vertex, where $a = 1$ and $b = 4$.

$$\begin{aligned} x &= -\frac{b}{2a} && \text{Plug-n-chug} \\ x &= -\frac{4}{2(1)} && \text{Simplify} \\ x &= -2 && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $f(-2)$.

$$\begin{aligned} f(x) &= x^2 + 4x + 3 && \text{Plug-n-chug} \\ f(-2) &= (-2)^2 + 4(-2) + 3 && \text{Evaluate} \\ f(-2) &= -1 && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $(-2, -1)$.

Property 3. *Find the y-intercept.* We can find the y-intercept by obtaining $f(0)$.

$$\begin{aligned} f(x) &= x^2 + 4x + 3 && \text{Plug-n-chug } x = 0 \\ f(0) &= 0^2 + 4(0) + 3 && \text{Evaluate} \\ f(0) &= 3 && \text{The } y\text{-intercept} \end{aligned}$$

Hence, the y-intercept is $(0, 3)$.

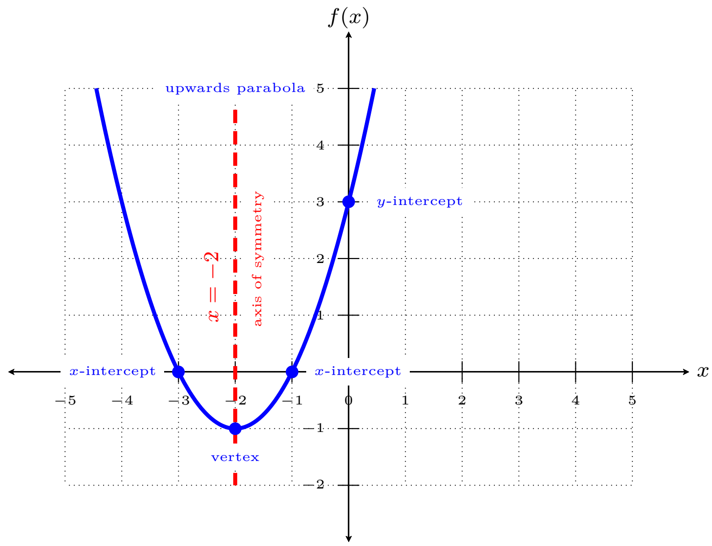
Property 4. *Find the x-intercepts.* We can find the x-intercept by obtaining where $f(x) = 0$.

$$\begin{aligned} f(x) &= x^2 + 4x + 3 && \text{Plug-n-chug } f(x) = 0 \\ 0 &= x^2 + 4x + 3 && \text{Factor} \\ 0 &= (x + 3)(x + 1) && \text{Apply the zero product rule} \\ x + 3 = 0 \quad \text{and} \quad x + 1 &= 0 && \text{Solve each equation} \\ x = -3 \quad \text{and} \quad x &= -1 && \text{The } x\text{-intercepts} \end{aligned}$$

Hence, the x-intercepts are $(-3, 0)$ and $(-1, 0)$.

Property 5. *Find the axis of symmetry.* We see from the vertex in **Property 2.** $x = -2$. Thus, the axis of symmetry is the vertical line $x = -2$.

Let's graph all the properties and label each property.



So, we verify that $f(x)$ is an upwards parabola with x-intercepts $(-3, 0)$ and $(-1, 0)$, y-intercept $(0, 3)$, vertex $(-2, -1)$, and axis of symmetry $x = -2$.

Example 11.23

Using the properties, graph $g(x) = -3x^2 + 12x - 9$.

Solution.

Property 1. *The direction of the parabola.* Since $a = -3$ and $a < 0$, then $g(x)$ is an downwards parabola.

Property 2. *Find the vertex.* We use the formula to find the vertex, where $a = -3$ and $b = 12$.

$$\begin{aligned} x &= -\frac{b}{2a} && \text{Plug-n-chug} \\ x &= -\frac{12}{2(-3)} && \text{Simplify} \\ x &= 2 && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $g(2)$.

$$\begin{aligned} g(x) &= -3x^2 + 12x - 9 && \text{Plug-n-chug} \\ g(2) &= -3(2)^2 + 12(2) - 9 && \text{Evaluate} \\ g(2) &= 3 && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $(2, 3)$.

Property 3. *Find the y-intercept.* We can find the y-intercept by obtaining $f(0)$.

$$\begin{aligned} g(x) &= -3x^2 + 12x - 9 && \text{Plug-n-chug } x = 0 \\ g(0) &= -30^2 + 12(0) - 9 && \text{Evaluate} \\ g(0) &= -9 && \text{The } y\text{-intercept} \end{aligned}$$

Hence, the y-intercept is $(0, -9)$.

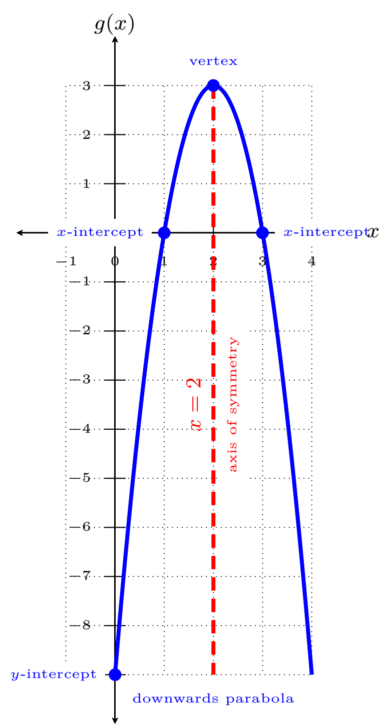
Property 4. *Find the x-intercepts.* We can find the x-intercept by obtaining where $g(x) = 0$.

$$\begin{aligned} g(x) &= -3x^2 + 12x - 9 && \text{Plug-n-chug } g(x) = 0 \\ 0 &= -3x^2 + 12x - 9 && \text{Factor a GCF of } -3 \\ 0 &= -3(x^2 - 4x + 3) && \text{Divide each side by } -3\text{, then factor} \\ 0 &= (x-3)(x-1) && \text{Apply the zero product rule} \\ x - 3 = 0 \quad \text{and} \quad x - 1 &= 0 && \text{Solve each equation} \\ x = 3 \quad \text{and} \quad x &= 1 && \text{The } x\text{-intercepts} \end{aligned}$$

Hence, the x-intercepts are $(3, 0)$ and $(1, 0)$.

Property 5. *Find the axis of symmetry.* We see from the vertex in **Property 2.** $x = 2$. Thus, the axis of symmetry is the vertical line $x = 2$.

Let's graph all the properties and label each property.



So, we verify that $g(x)$ is an downwards parabola with x-intercepts $(3, 0)$ and $(1, 0)$, y-intercept $(0, -9)$, vertex $(2, 3)$, and axis of symmetry $x = 2$.

11.4.3 Graph quadratic functions by transformations

Definition

A quadratic function in ***vertex form*** is given as

$$f(x) = a(x-h)^2 + k,$$

where the domain consists of all real numbers and (h, k) is the vertex.

Recall. In the chapter with rational functions, we graphed rational functions with horizontal and vertical shift. Let's take this one step further, and graph quadratic functions with not only horizontal and vertical shifts, but also with a stretch or compression.

Transformations of Quadratic Functions

Given $f(x)$ is the quadratic function in vertex form

$$f(x) = a(x-h)^2 + k,$$

horizontal and vertical shifts, and vertical stretches or compressions of $f(x)$ are described below:

	$\boldsymbol{f(x-h)}$	$\boldsymbol{f(x) \pm k}$	$\boldsymbol{af(x)}$
Transformation	Horizontal shift	Vertical shift	Vertical stretch or compression
Units	$h > 0$: Shift h units to the right $h < 0$: Shift h units to the left	$k > 0$: Shift k units upwards $k < 0$: Shift k units downwards	$\lvert a\rvert > 1$: Multiply all outputs by a; *vertical stretch* $0 < \lvert a\rvert < 1$: Multiply all outputs by a; *vertical compression*

Recall from the previous subsection, if $a > 0$, the parabola is upward and if $a < 0$, the parabola is downward.

Example 11.24

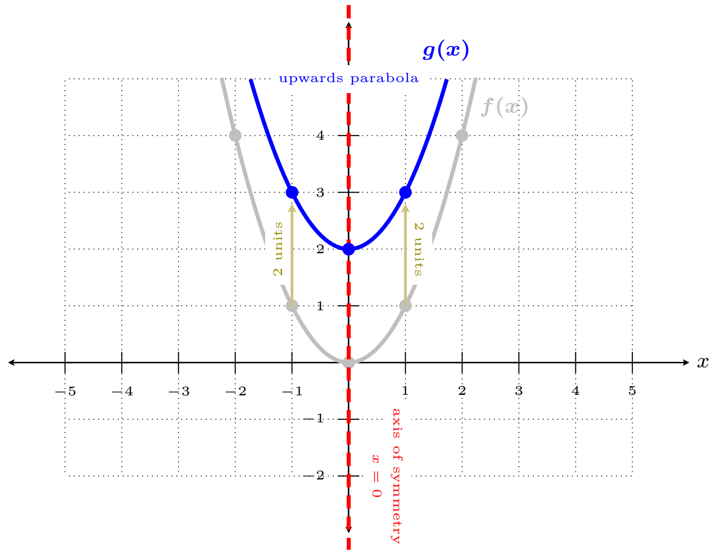
Using the library function $f(x) = x^2$, graph $g(x) = x^2 + 2$.

Solution.

We start by noticing we are adding 2 to $f(x)$, i.e., $g(x) = f(x) + 2$:

$$g(x) = x^2 + 2$$
$$g(x) = f(x) + 2$$

This means, from the table, $g(x)$ has a vertical shift by 2 units upward. Let's start with $f(x) = x^2$, and then shift $f(x)$ 2 units upward to obtain $g(x)$:



We can see that the blue solid graph is $g(x)$, where $g(x)$ is an upward parabola, the axis of symmetry is $x = 0$. Notice, all points on $f(x)$ shifted upwards by 2 units. We can use the well-defined points of the library function $f(x) = x^2$ to transform into $g(x)$.

Example 11.25

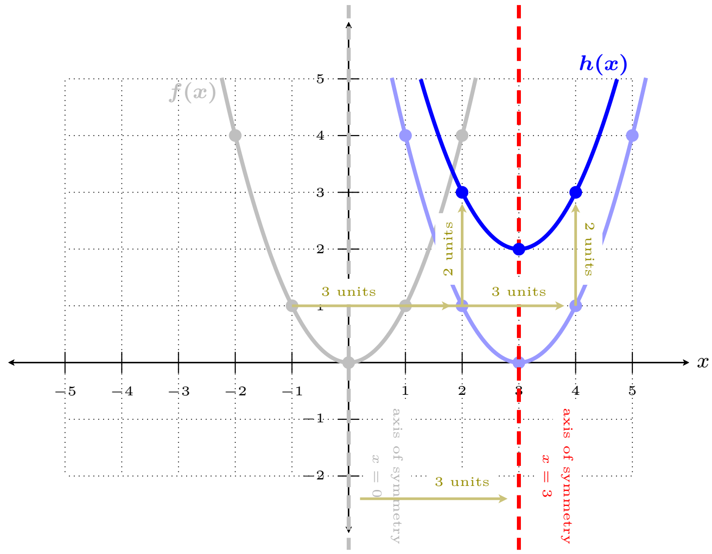
Using the library function $f(x) = x^2$, graph $h(x) = (x - 3)^2 + 2$.

Solution.

We start by noticing we are adding 2 to $f(x)$ and subtracting 3 from the input x, i.e., $h(x) = f(x - 3) + 2$:

$$h(x) = (x - 3)^2 + 2$$
$$h(x) = f(x - 3) + 2$$

This means, from the table, $h(x)$ has a horizontal shift to the right by 3 units, and vertical shift by 2 units upward. Let's start with $f(x) = x^2$, and then shift $f(x)$ 3 units to the right, and 2 units upward to obtain $h(x)$:



We can see that the blue solid graph is $h(x)$, where $h(x)$ is an upward parabola, the axis of symmetry is $x = 3$. Notice, all points on $f(x)$ shifted right by 3 units and upwards by 2 units. We can use the well-defined points of the library function $f(x) = x^2$ to transform into $h(x)$.

Example 11.26

Using the library function $f(x) = x^2$, graph $k(x) = -2(x + 1)^2 - 3$.

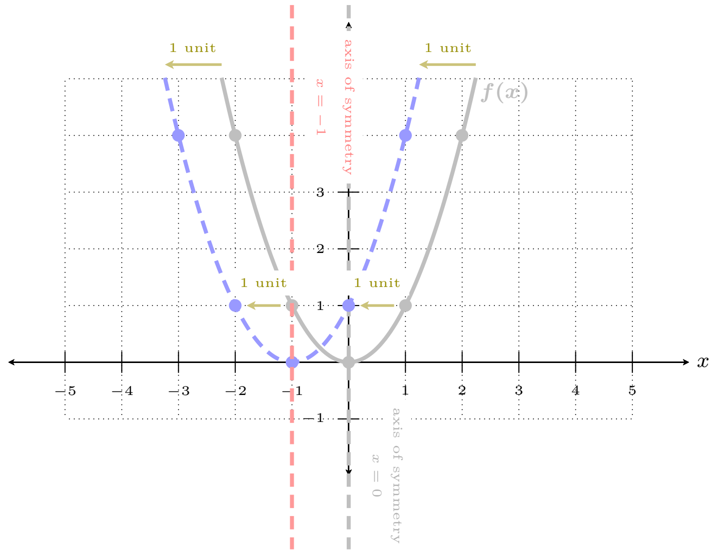
Solution.

We start by noticing we are subtracting 3 from $f(x)$, vertically stretching by a factor of -2, and adding 1 to the input x, i.e., $k(x) = -2 \cdot f(x + 1) - 3$:

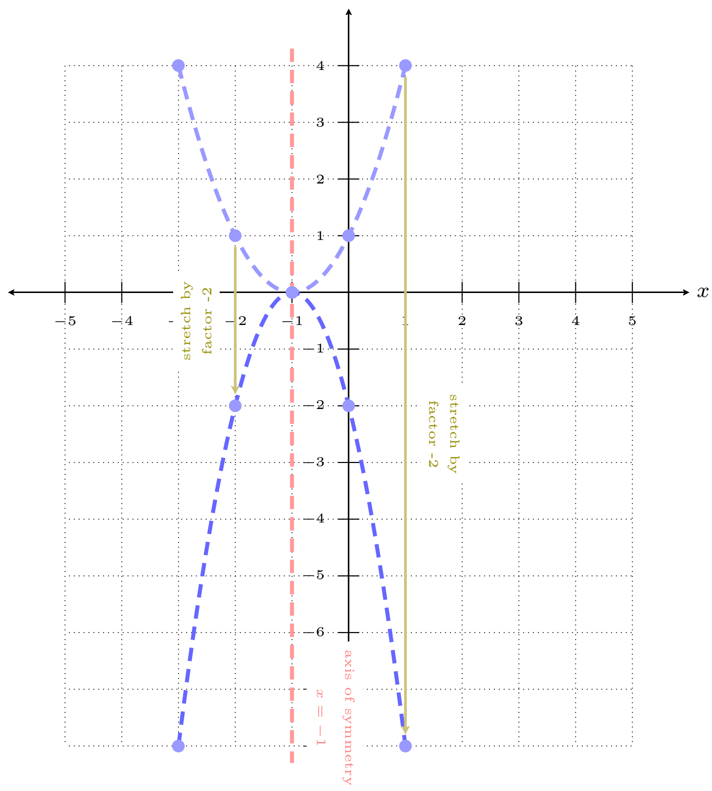
$$k(x) = -2(x + 1)^2 - 3$$
$$k(x) = -2 \cdot f(x + 1) - 3$$

This means, from the table, $k(x)$ has a horizontal shift to the left by 1 unit, a vertical stretch by a factor of -2, and a vertical shift by 3 units downward. Let's start with $f(x) = x^2$, and then apply these transformations to obtain $k(x)$. Since there are three transformations, it is best we split this up into three steps.

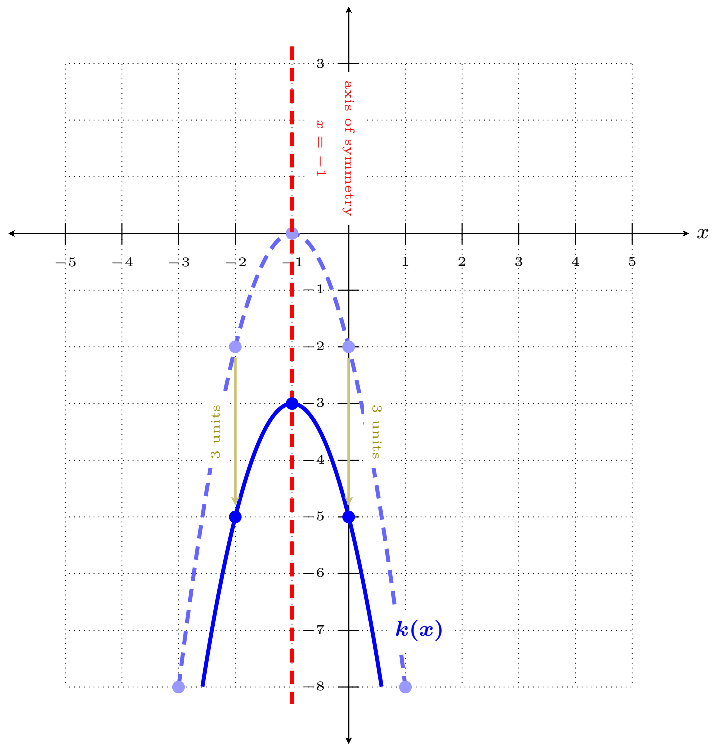
Step 1. Graph the library function, $f(x) = x^2$, and apply the horizontal shift: $f(x + 1)$.



Step 2. Graph $f(x+1)$ from **Step 1.**, and apply the vertical stretch, i.e., multiply the y-coordinates of the well-defined ordered pairs by -2: $-2 \cdot f(x+1)$



Step 3. Graph $-2 \cdot f(x+1)$ from **Step 2.**, and apply the vertical shift: $-2 \cdot f(x+1) - 3$



We can see that the blue solid graph is $k(x)$, where $k(x)$ is an downward parabola, the axis of symmetry is $x = -1$. Notice, all points on $f(x)$ shifted left by 1 unit, stretched by a factor of -2, and shifted downwards by 3 units.

Note

Notice with three transformations, it can get tedious and tricky. The best route to apply multiple transformations is to follow order of operations, e.g., first parenthesis, multiplication/division, and then addition/subtraction. With transformations, this translates to

Step 1. Apply the horizontal shift

Step 2. Vertically stretch or compress the function

Step 3. Lastly, apply the vertical shift

A way to remember the order in which we apply the transformations is *hak*: first the h, then a, lastly, k.

11.4.4 Graph Quadratic Functions Homework

Graph the quadratic function using the properties. Be sure to label your graph with all properties.

1. $f(x) = x^2 - 2x - 8$

2. $f(x) = 2x^2 - 12x + 10$

3. $f(x) = -2x^2 + 12x - 18$

4. $f(x) = -3x^2 + 24x - 45$

5. $f(x) = -x^2 + 4x + 5$

6. $f(x) = -x^2 + 6x - 5$

7. $f(x) = -2x^2 + 16x - 24$

8. $f(x) = 3x^2 + 12x + 9$

9. $f(x) = 5x^2 - 40x + 75$

10. $f(x) = -5x^2 - 60x - 175$

11. $f(x) = x^2 - 2x - 3$

12. $f(x) = 2x^2 - 12x + 16$

13. $f(x) = -2x^2 + 12x - 10$

14. $f(x) = -3x^2 + 12x - 9$

15. $f(x) = -x^2 + 4x - 3$

16. $f(x) = -2x^2 + 16x - 30$

17. $f(x) = 2x^2 + 4x - 6$

18. $f(x) = 5x^2 + 30x + 45$

19. $f(x) = 5x^2 + 20x + 15$

20. $f(x) = -5x^2 + 20x - 15$

Starting with the library function $y = x^2$, state the function, $f(x)$, given its transformation(s).

21. vertically stretched by a factor of 3 and shifted right 1

22. vertically stretched by a factor of -2, and shifted left 3

23. vertically compressed by a factor of $\frac{1}{3}$

24. vertically stretched by a factor of 2 and shifted right 4

25. vertically stretched by a factor of 4, shifted left 4

26. vertically compressed by a factor of $-\frac{1}{2}$ and shifted upward by 3 units

27. vertically stretched by a factor of 3 and shifted down 4

28. vertically stretched by a factor of 2, shifted right 3, and shifted up 1

Starting with the library function $y = x^2$, graph the function using transformations.

29. $g(x) = -\frac{1}{2}(x+7)^2$

30. $g(x) = 2(x-1)^2 - 2$

31. $y = \frac{1}{5}(x-2)^2$

32. $f(x) = \frac{1}{2}(x+2)^2 + 9$

33. $f(x) = 2(x+4)^2 - 5$

34. $f(x) = -2(x-4)^2 + 7$

35. $g(x) = 2(x-3)^2 - 2$

36. $g(x) = -\frac{1}{2}(x+5)^2$

37. $f(x) = \frac{1}{2}(x+4)^2 + 8$

38. $f(x) = -2(x-8)^2 + 7$

11.5 Quadratic inequalities

We can think about an earlier chapter where we obtained solutions for linear inequalities where the solutions are intervals of numbers. With quadratic inequalities, it is similar to linear inequalities, but, now, instead of lines, we have parabolas. Hence, we have two methods for solving quadratic inequalities: graphing or algebraically.

11.5.1 Solving quadratic inequalities by graphing

We can use the graph of a quadratic function, a parabola, to obtain the solutions to a quadratic inequality. We easily graph the function, then determine where the values are true for the inequality.

Note

We can use the zeros (or roots) of a graph of a quadratic equation to solve for quadratic inequalities. Recall, the zeros (or roots) of a graph are the x-intercepts, i.e., $y = 0$.

Example 11.27

Solve $x^2 - 3x - 4 \geq 0$ by using the graphing method.

Solution.

We first begin by graphing $x^2 - 3x - 4$ as we would in the previous section. Recall, the x-coordinate of the vertex is $-\dfrac{b}{2a}$. Hence,

$$-\frac{b}{2a} = -\frac{-3}{2(1)} = \frac{3}{2} = 1.5$$

The y-coordinate of the vertex is $x^2 - 3x - 4$ evaluated for $x = 1.5$:

$$(1.5)^2 - 3(1.5) - 4 = -6.25$$

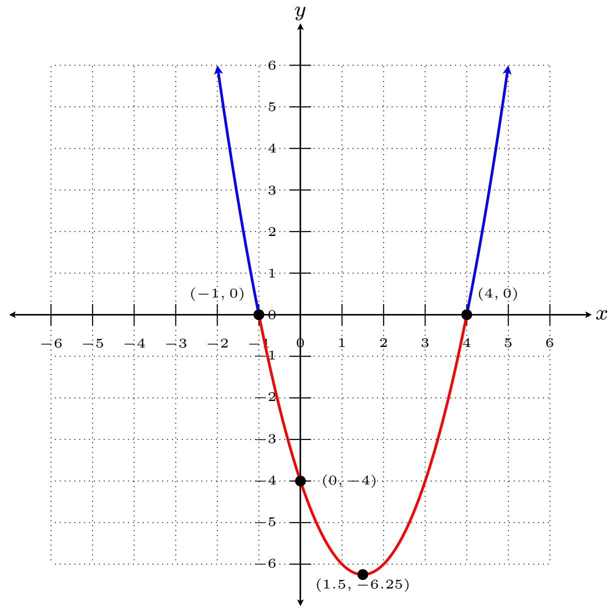
The vertext is at $(1.5, -6.25)$. The y-intercept is when $x = 0$:

$$0^2 - 3(0) - 4 = -4$$

and the x-intercepts are when $x^2 - 3x - 4 = 0$. Solving for the roots we get

$$\begin{aligned} x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ x - 4 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = 4 \quad &\text{or} \quad x = -1 \end{aligned}$$

Thus, the intercepts are $(0, -4)$, $(4, 0)$, and $(-1, 0)$ with a vertex at $(1.5, -6.25)$. Now we draw the graph:



Given $x^2 - 3x - 4 \geq 0$ says we are specifically looking for the values of $x^2 - 3x - 4$ in which are greater than or equal to zero. In other words, we are looking for all y-values that are *above* the x-axis because that is where all the y-values are positive. Looking at the graph above, these are all the values on the blue parts of the graph. Thus, using the graphing method, the solution is $(-\infty, -1] \cup [4, \infty)$. Note, we use brackets since it is $\geq$.

Notice, we ignore the red part of the parabola since this is where $x^2 - 3x - 4$ is negative, i.e., all the y-values are negative, because we were only looking for when $x^2 - 3x - 4 \geq 0$, the parts of the graph above the x-axis.

11.5.2 Solving quadratic inequalities algebraically

The next method to solving quadratic inequalities is algebraically. Hence, there is a little more work involved since the method is algebraic. We must follow an order of steps to obtain the correct solution. The good news is that these two methods are similar where we always use the x-intercepts to determine the intervals.

Steps to solving quadratic inequalities

Step 1. Rewrite the inequality so that $ax^2 + bx + c$ is on one side and zero is on the other.

Step 2. Determine where the *inequality is zero* using any method appropriate.

Step 3. Use the x-values obtained in the previous step to label on a number line.

Step 4. Take test values to observe where the inequality is true.

- If the inequality is < 0 or ≤ 0, then the inequality is true where the test values are negative.
- If the inequality is > 0 or ≥ 0, then the inequality is true where the test values are positive.

Step 5. Write the solution in interval notation.

Example 11.28

Solve algebraically: $-x^2 + 8 > 2x$

Solution.

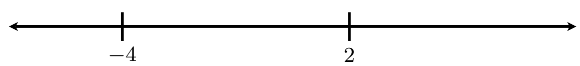
Step 1. We rewrite $-x^2 + 8 > 2x$ so that zero is on one side:

$$-x^2 - 2x + 8 > 0$$

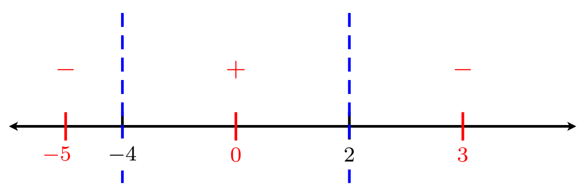
Step 2. We set the left side equal to zero to obtain the roots (or zeros):

$$\begin{aligned}
-x^2 - 2x + 8 &= 0 \\
x^2 + 2x - 8 &= 0 \\
(x+4)(x-2) &= 0 \\
x + 4 = 0 \quad &\text{or} \quad x - 2 = 0 \\
x = -4 \quad &\text{or} \quad x = 2
\end{aligned}$$

Step 3. Label -4 and 2 on a blank number line:



Step 4. We take test values on each side of -4 and 2. Let's choose fairly easy numbers such as -5, 0, and 3. We plug these numbers into $-x^2 - 2x + 8$ and determine whether the value is positive or negative:



$$\begin{aligned}
-(-5)^2 - 2(-5) + 8 = -7 < 0 \quad &\implies \text{letting } x = -5 \\
-(0)^2 - 2(0) + 8 = 8 > 0 \quad &\implies \text{letting } x = 0 \\
-(3)^2 - 2(3) + 8 = -7 < 0 \quad &\implies \text{letting } x = 3
\end{aligned}$$

Step 5. Since $-x^2 - 2x + 8 > 0$ (from **Step 1.**), then we are looking for where the test values are positive. Looking at the number line above, we see these are the values in between -4 and 2. Thus, the solution is $(-4, 2)$. Note, we use parenthesis since the inequality symbol is $>$.

11.5.3 Quadratic Inequalities Homework

Solve the inequality. Write the solution in interval notation.

1. $x^2 - 9x + 18 > 0$

2. $x^2 - 2x - 24 \leq 0$

3. $x^2 - 2x - 3 < 0$

4. $x^2 + 10x + 24 \geq 0$

5. $x^2 - 4x + 4 > 0$

6. $x^2 + 2x \geq 8$

7. $x^2 - 4x \leq -3$

8. $3x^2 + 7x - 20 \leq 0$

9. $4x^2 + 11x - 20 \geq 0$

10. $3x^2 + 2x - 1 < 0$

11. $-4x^2 + 7x \geq 0$

12. $x^2 + 6x \geq 0$

13. $x^2 - 12x + 36 < 0$

14. $x^2 - 2x + 1 \geq 0$

11.6 Applications with quadratic functions

There are many applications involving quadratic functions that it is almost challenging to pick the few functions to discuss for this section. Yet, we only choose a few applications that are common in most algebra classes. First, we must start with defining the ***extreme value*** of a quadratic function.

11.6.1 Find the extreme value

Definition

The ***extreme value*** of a quadratic function $f(x)$ is either the *minimum* or *maximum* value of the quadratic function $f(x)$. The extreme value is given as

$$f\left(-\frac{b}{2a}\right)$$

The *minimum* value is located at the lowest point of an upwards parabola and the *maximum* value is located at the highest point of a downwards parabola.

✎ **Note**

Notice in the definition that the extreme value is the y-coordinate of the vertex. Recall, the vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

and we can see the y-coordinate is just the extreme value. Hence, when we use the words vertex, minimum or maximum value, or extreme value, they are all associated with the vertex of a parabola.

Example 11.29

Find the vertex and the extreme value of the function $f(x) = x^2 - 4x + 5$.

Solution.

Notice $a = 1$, which means $a > 0$. Hence, $f(x)$ is an upwards parabola and, from the definition, we expect $f(x)$ to have a minimum value. Let's use the formula to find the vertex, where $a = 1$ and $b = -4$.

$$\begin{aligned} x &= -\frac{b}{2a} && \text{Plug-n-chug} \\ x &= -\frac{-4}{2(1)} && \text{Simplify} \\ x &= 2 && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $f(2)$.

$$\begin{aligned} f(x) &= x^2 - 4x + 5 && \text{Plug-n-chug} \\ f(2) &= (2)^2 - 4(2) + 5 && \text{Evaluate} \\ f(2) &= 1 && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $(2, 1)$. Next, we find the extreme value of $f(x)$. From the vertex calculation, we see $f\left(-\frac{b}{2a}\right) = f(2) = 1$. Thus, the extreme value is 1, the y-coordinate of the vertex.

Example 11.30

Find the vertex and the extreme value of the function $q(n) = -3n^2 - 5n + 3$.

Solution.

Notice $a = -3$, which means $a < 0$. Hence, $q(n)$ is an downwards parabola and, from the definition, we expect $q(n)$ to have a maximum value. Let's use the formula to find the vertex, where $a = -3$ and $b = -5$.

$$\begin{aligned} n &= -\frac{b}{2a} && \text{Plug-n-chug} \\ n &= -\frac{-5}{2(-3)} && \text{Simplify} \\ n &= -\frac{5}{6} && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $q\left(-\frac{5}{6}\right)$.

$$\begin{aligned} q(n) &= -3n^2 - 5n + 3 && \text{Plug-n-chug} \\ q\left(-\frac{5}{6}\right) &= -3\left(-\frac{5}{6}\right)^2 - 5\left(-\frac{5}{6}\right) + 3 && \text{Evaluate} \\ q\left(-\frac{5}{6}\right) &= \frac{61}{12} && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $\left(-\frac{5}{6}, \frac{61}{12}\right)$. Next, we find the extreme value of $q(n)$. From the vertex calculation, we see $q\left(-\frac{b}{2a}\right) = q\left(-\frac{5}{6}\right) = \frac{61}{12}$. Thus, the extreme value is $\frac{61}{12}$, the y-coordinate of the vertex.

It may seem a little redundant to find the vertex and the extreme value, but, remember, the goal is to apply this concept to real-world applications with quadratic functions. Let's take a look at a few applications where we find the extreme value in context of a real-world model.

11.6.2 Projectile motion

Example 11.31

A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h(t) = -4.9t^2 + 46t + 157$. At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round the answers to 2 decimal places.*

Solution.

When we read the word *maximum*, we should think about the vertex of $h(t)$. Since we need to find the time in which the maximum height occurs, then we can find the x-coordinate of the vertex.

$$t = -\frac{b}{2a}$$
$$t = -\frac{46}{2(-4.9)}$$
$$t \approx 4.69$$

Thus, the maximum height occurs after 4.69 seconds. Next, we find the height of the rocket when it reaches its maximum height above the water. Since we need to find the maximum height, then we need to find the y-coordinate of the vertex, or $h(4.69)$.

$$h(4.69) = -4.9(4.69)^2 + 46(4.69) + 157 \approx 264.96$$

Thus, the maximum height of the rocket is 264.96 meters after 4.69 seconds the rocket is launched.

11.6.3 Revenue and cost functions

Example 11.32

The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function $R(x) = -6x^2 + 108x$. What is the maximum revenue?

Solution.

To find the maximum revenue, we need to find $R\left(-\frac{b}{2a}\right)$, where $a = -6$ and $b = 108$. Let's plug-n-chug this into $R(x)$ to find the maximum revenue.

$$R(x) = -6x^2 + 108x$$
$$R\left(-\frac{b}{2a}\right) = -6\left(-\frac{b}{2a}\right)^2 + 108\left(-\frac{b}{2a}\right)$$
$$R\left(-\frac{108}{2(-6)}\right) = -6\left(-\frac{108}{2(-6)}\right)^2 + 108\left(-\frac{108}{2(-6)}\right)$$
$$R\left(-\frac{108}{2(-6)}\right) = 486$$

Thus, the maximum revenue is \$486.

Example 11.33

The cost, $C(x)$, of producing x *Totally Cool Coolers* is modeled by the function $C(x) = 0.005x^2 - 0.3x + 17$. What is the minimum cost?

Solution.

To find the minimum cost, we need to find $C\left(-\frac{b}{2a}\right)$, where $a = 0.005$ and $b = -0.3$. Let's plug-n-chug this into $C(x)$ to find the minimum cost.

$$\begin{aligned}
C(x) &= 0.005x^2 - 0.3x + 17 \\
C\left(-\frac{b}{2a}\right) &= 0.005\left(-\frac{b}{2a}\right)^2 - 0.3\left(-\frac{b}{2a}\right) + 17 \\
C\left(-\frac{-0.3}{2(0.005)}\right) &= 0.005\left(-\frac{-0.3}{2(0.005)}\right)^2 - 0.3\left(-\frac{-0.3}{2(0.005)}\right) + 17 \\
C\left(-\frac{-0.3}{2(0.005)}\right) &= 12.5
\end{aligned}$$

Thus, the minimum cost is \$12.50.

Note

From all the examples, we see the variety of methods in obtaining the extreme value of a quadratic function. We can either graph the function, find each coordinate of the vertex, or directly calculate the extreme value. It is at the discretion of the student to use any method. However, directly calculating the extreme value is recommended when only the extreme value is needed.

11.6.4 Applications with Quadratic Functions Homework

1. Find the vertex and the extreme value of the function

$$f(x) = 2x^2 - 5x - 4$$

What is the vertex? What is the extreme value?

2. Find the vertex and the extreme value of the function

$$f(x) = 2x^2 - 2x - 5$$

What is the vertex? What is the extreme value?

3. Find the vertex and the extreme value of the function

$$f(x) = x^2 + 2x + 2$$

What is the vertex? What is the extreme value?

4. A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by

$$h(t) = -4.9t^2 + 271t + 150$$

At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round your answer to 2 decimal places.*

5. A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by

$$h(t) = -4.9t^2 + 190t + 395$$

At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round your answer to 2 decimal places.*

6. A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by

$$h(t) = -4.9t^2 + 223t + 129$$

At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round your answer to 2 decimal places.*

7. The cost, $C(x)$, of producing x *Totally Cool Coolers* is modeled by the function

$$C(x) = 0.005x^2 - 0.25x + 12$$

How many coolers need to be produced and sold in order to minimize the cost? What is the cost?

8. The cost, $C(x)$, of producing x *Totally Cool Coolers* is modeled by the function

$$C(x) = 0.005x^2 - 0.45x + 25$$

How many coolers need to be produced and sold in order to minimize the cost? What is the cost?

9. The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function

$$R(x) = -5x^2 + 105x$$

How many hearing aids need to be produced and sold in order to maximize the revenue? What is the revenue?

10. The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function

$$R(x) = -2x^2 + 62x$$

How many hearing aids need to be produced and sold in order to maximize the revenue? What is the revenue?

11. The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function

$$R(x) = -4x^2 + 76x$$

How many hearing aids need to be produced and sold in order to maximize the revenue? What is the revenue?

11.7 Quadratic Equations and Applications: Answers to the Homework Exercises

Square Root Property

1. $7, -1$
3. $9, 5$
5. $4, -14$
7. $-9 \pm \sqrt{37}$
9. $2 \pm \sqrt{70}$
11. $-4 \pm 3\sqrt{7}$
13. $\frac{2}{9}, -\frac{4}{9}$
15. $\frac{11}{3}, \frac{1}{3}$
17. $-6 \pm 5\sqrt{3}$
19. $-\frac{1}{2}$
21. $4, -10$
23. $\pm 2\sqrt{2}$
25. $-9 \pm 2\sqrt{7}$
27. $-8 \pm 3\sqrt{5}$
29. $0, -3$
31. $-\frac{2}{7}, -\frac{10}{7}$
33. $\frac{7}{4}, \frac{3}{4}$

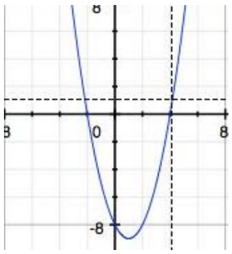
Completing the Square

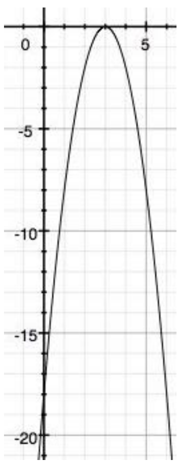
1. 225; $(x-15)^2$
3. $\frac{225}{4}$; $\left(x-\frac{15}{2}\right)^2$
5. 144; $(a-12)^2$
7. $\frac{1}{324}$; $\left(r-\frac{1}{18}\right)^2$
9. $11, 5$
11. $-1 \pm i\sqrt{\frac{19}{2}}$
13. $-5+\sqrt{86}, -5-\sqrt{86}$
15. $-1 \pm i\sqrt{21}$
17. $-5 \pm 2i$
19. $1 \pm i\sqrt{2}$
21. $4 \pm i\sqrt{39}$
23. $-1 \pm i\sqrt{\frac{43}{6}}$
25. $-\frac{5}{2} \pm \frac{i\sqrt{87}}{2}$
27. $\frac{7}{2} \pm \frac{i\sqrt{139}}{2}$
29. $\frac{9+\sqrt{21}}{2}, \frac{9-\sqrt{21}}{2}$
31. $1, -\frac{5}{2}$
33. $4+2\sqrt{7}, 4-2\sqrt{7}$
35. $1 \pm 2i\sqrt{\frac{11}{3}}$
37. $8+2\sqrt{29}, 8-2\sqrt{29}$
39. $1, -3$
41. $7+\sqrt{85}, 7-\sqrt{85}$
43. $x = 1 \pm i\sqrt{\frac{21}{5}}$
45. $-1, -7$
47. $-2 - i\sqrt{\frac{37}{3}}$
49. $\frac{-7+\sqrt{181}}{2}, \frac{-7-\sqrt{181}}{2}$
51. $\frac{5}{2} \pm 3i\sqrt{7}$
53. $\frac{1}{2} \pm i\sqrt{163}$
55. $8, 7$

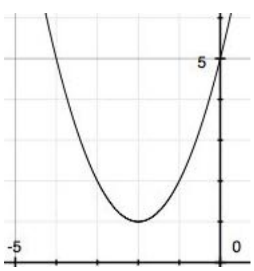
Quadratic Formula

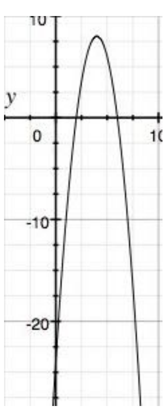
1. $\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}$
3. $\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}$
5. $3, -3$
7. $\frac{-3 \pm i\sqrt{55}}{4}$
9. $\frac{-3+\sqrt{401}}{14}, \frac{-3-\sqrt{401}}{14}$
11. $\frac{-1 \pm i\sqrt{3}}{2}$
13. $\frac{7+3\sqrt{21}}{10}, \frac{7-3\sqrt{21}}{10}$
15. $-1, -\frac{3}{2}$
17. $4, -9$
19. $\frac{-3+\sqrt{345}}{14}, \frac{-3-\sqrt{345}}{14}$
21. $\pm \frac{i\sqrt{6}}{3}$
23. $\frac{-1 \pm i\sqrt{29}}{5}$
25. $\pm \frac{i\sqrt{2}}{2}$
27. $\frac{3 \pm i\sqrt{159}}{12}$
29. $\frac{-5+\sqrt{137}}{8}, \frac{-5-\sqrt{137}}{8}$
31. $3, -\frac{1}{3}$
33. $\frac{-5+\sqrt{337}}{12}, \frac{-5-\sqrt{337}}{12}$
35. $2\sqrt{2}, -2\sqrt{2}$
37. $\frac{3 \pm 3i\sqrt{5}}{7}$
39. $\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}$

Graph Quadratic Functions

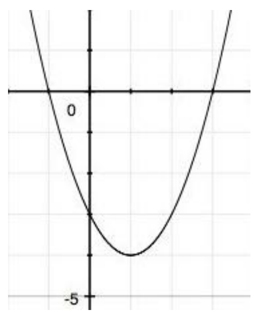
1. 

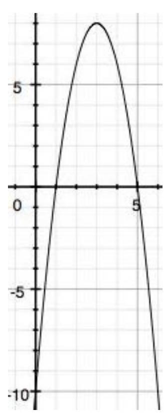
3. 

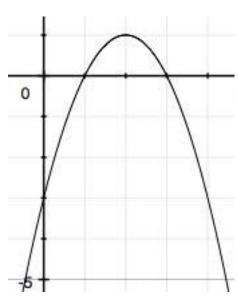
5. 

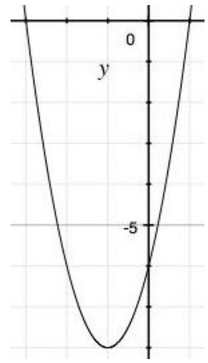
7. 

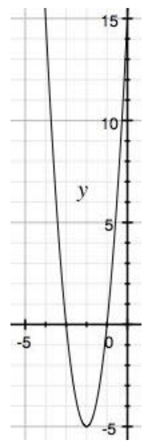
9. 

11. 

13. 

15. 

17. 

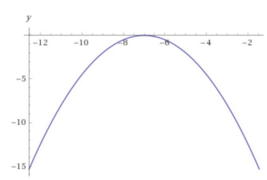
19. 

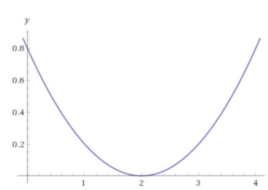
21. $f(x) = 3(x-1)^2$

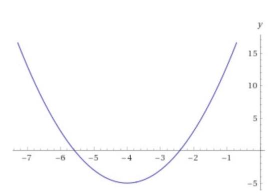
23. $f(x) = \frac{1}{3}x^2$

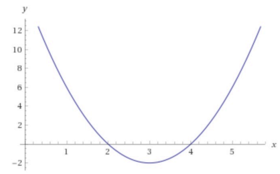
25. $f(x) = 4(x+4)^2$

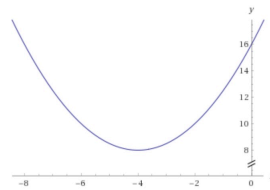
27. $f(x) = 3x^2 - 4$

29. 

31. 

33. 

35. 

37. 

Quadratic Inequalities

1. $(-\infty, 3) \cup (6, \infty)$
2. $[-4, 6]$
3. $(-1, 3)$
4. $(-\infty, -6] \cup [-4, \infty)$
5. $(-\infty, 2) \cup (2, \infty)$
6. $(-\infty, -4] \cup [2, \infty)$
7. $[1, 3]$
8. $\left[-4, \frac{5}{3}\right]$
9. $(-\infty, -4] \cup \left(\frac{5}{4}, \infty\right)$
10. $\left(-1, \frac{1}{3}\right)$
11. $\left[0, \frac{7}{4}\right]$
12. $(-\infty, -6] \cup [0, \infty)$
13. No Solution
14. All real numbers

Applications with Quadratic Functions

1. $\left(\frac{5}{4}, -\frac{57}{8}\right); -\frac{57}{8}$
2. $\left(\frac{1}{2}, -\frac{11}{2}\right); -\frac{11}{2}$
3. $(-1, 1)$; 1
4. 27.65; 3,896.99
5. 19.39; 2,236.84
6. 22.76; 2,666.19
7. 25; 8.88
8. 45; 14.88
9. 11; 551.25 or 550
10. 16; 480.5 or 480
11. 10; 361 or 360

Chapter 12

Exponential and Logarithmic Functions

Chapter Objectives

By the end of this chapter, the student should be able to

- Find the inverse of a function (both equation and graph)
- Apply the properties of logarithms
- Solve exponential and logarithmic equations
- Graph exponential and logarithmic functions
- Determine the domain of exponential and logarithmic functions
- Solve applications using exponential and logarithmic functions

As our study of algebra advances, we begin to study more complex functions. One pair of functions that have a distinct relationship with each other is *exponential* and *logarithmic* functions. First, we take a look at a special relationship between certain functions, then move into exponential and logarithmic functions.

12.1 Inverse functions

12.1.1 One-to-one functions

Let's begin by discussing the relationship of one-to-one. One-to-one functions are special functions where all the inputs and outputs of a function are distinct, i.e., none of the x or y coordinates repeat.

Definition

A function is ***one-to-one*** if any two different inputs in the domain correspond to two different outputs in the range, i.e., $f(x_1) \neq f(x_2)$ for any x_1 and x_2.

Another way of determining one-to-one is to make sure all x and all y values are different.

Example 12.1

Determine if the following relation is one-to-one.

$$\{(3,-5),(2,-1),(1,0),(0,7),(-1,8)\}$$

Solution.

We first look at all the inputs and outputs. By the definition, we need to make sure no x or y values repeat, i.e., all x and y coordinates are unique. We have

$$x - \text{values} = 3, 2, 1, 0, -1$$

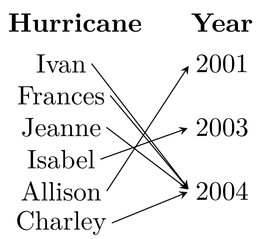
and

$$y - \text{values} = -5, -1, 0, 7, 8$$

Hence, none of the coordinates repeat, which means this relation is one-to-one.

Example 12.2

Determine if the following relation is one-to-one.



Solution.

We first look at all the inputs and outputs. Let's assume the name of the hurricane is the input and the year the hurricane took place is the output. By the definition, we need to make sure no x or y values repeat, i.e., all x and y coordinates are unique. We have

$$x - \text{values} = \text{Ivan, Frances, Jeanne, Isabel, Allison, Charley}$$

and

$$y - \text{values} = 2004, 2004, 2004, 2003, 2001, 2004$$

Hence, none of the x coordinates repeat, but the year 2004 repeats itself 4 times in the y-coordinates, which means this relation is *not* one-to-one.

World Note

The notation used for functions was first introduced by the great Swiss mathematician, *Leonhard Euler*, in the 18$^{\text{th}}$ century.

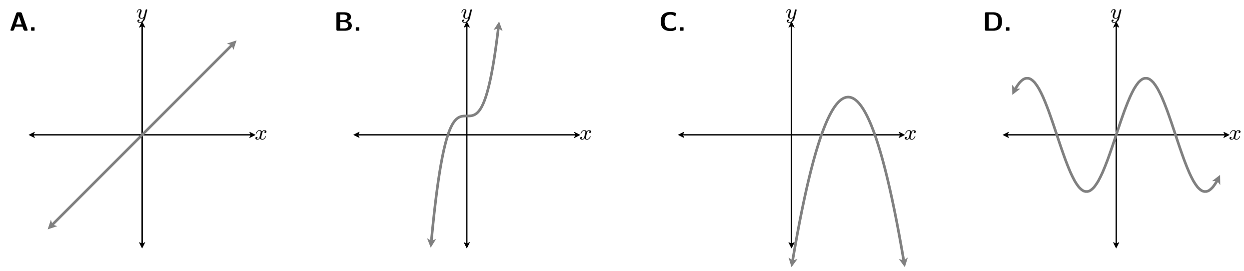
There is a graphical way to determine whether a given graph of a function is one-to-one and that is by the horizontal line test. As we used the vertical line test to determine whether a graph is a function, we use the horizontal line test to determine whether a graph of a function is one-to-one.

Horizontal line test

If every *horizontal line* intersects the graph of a function f at most one point, then f is one-to one.

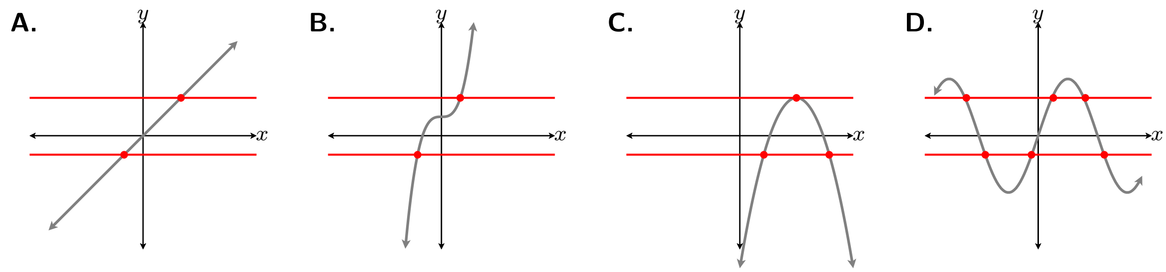
Example 12.3

Determine which graphs of the functions below are one-to one.



Solution.

Let's start by drawing horizontal lines throughout each graph and determine whether the line intersects the graph more than once. Now, recall, it is given that these graphs are all functions. Meaning, we assume all four of these functions have passed the vertical line test. We just need to see whether these functions are one-to-one by applying the horizontal line test.



Looking at **A.** and **B.**, we see that the horizontal lines intersect the graphs only once, passing the horizontal line test. If we take a look at **C.**, the top line intersects the graph once because it is only intersecting at the parabola's vertex, but looking at the bottom line, we see the horizontal line intersects the parabola two times. Hence, **C.** doesn't pass the horizontal line test. Lastly, **D.** has both of its horizontal lines intersecting the graph more than once and resulting in failing the horizontal line test. Thus, graphs **A.** and **B.** both pass the horizontal line test and are one-to-one functions.

☠ ***Warning!***

Be sure to draw complete horizontal lines, from left to right, filling the grid, and more than one. It is easy to draw a line and stop midway to conclude the graph passes the horizontal line test, like in **C.**. However, for the validity of the horizontal line test, we must draw complete horizontal lines, left to right, filling the grid, and more than one.

12.1.2 A function and its inverse

Definition

If $f(x)$ is one-to-one, we call $f(x)$ an ***invertible function*** with ordered pairs (a, b). The inverse function, $f^{-1}(x)$, is the set of ordered pairs (b, a), i.e., y-coordinates and x-coordinates switch.

Example 12.4

Find the inverse of the one-to-one function

$$\{(3,-1),(2,7),(1,-4),(0,8),(-1,5)\}$$

State the domain and range of the inverse function.

Solution.

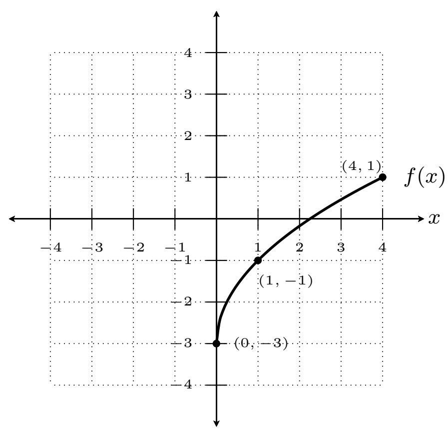
To find the inverse of a given one-to-one function, we need to identify all x and y coordinates and reverse them, i.e., by the definition, y-coordinates and x-coordinates switch. Let $f(x) = \{(3,-1),(2,7),(1,-4),(0,8),(-1,5)\}$. Then

$$f^{-1}(x) = \{(-1,3),(7,2),(-4,1),(8,0),(5,-1)\}$$

The domain of $f^{-1}(x)$ is $\{-4,-1,5,7,8\}$ and the range is $\{-1,0,1,2,3\}$.

Example 12.5

Draw the inverse function of the given one-to-one function.



Solution.

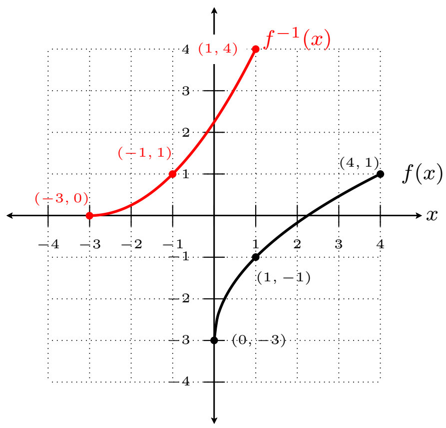
Using the same rationale as we did for Example 12.4, we can take the well-defined ordered pairs on the graph of $f(x)$ and switch the x and y-coordinates. Let's place the ordered-pairs on a table:

x	$f(x)$
0	-3
1	-1
4	1

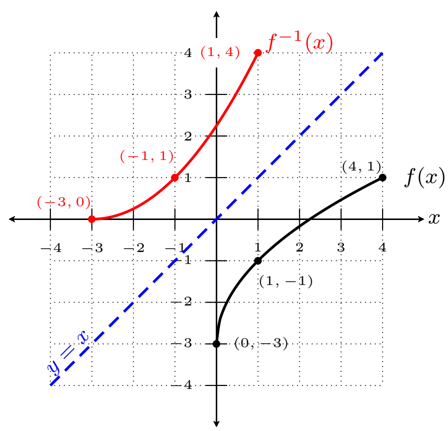
this impies that $f^{-1}(x)$ is

x	$f^{-1}(x)$
-3	0
-1	1
1	4

Notice, all we did was switch the x and y-coordinates from the first table to obtain three well-defined ordered pairs on $f^{-1}(x)$. Let's graph these points and connect them with a nice smooth curve:



We can see from Example 12.5, that the idea behind invertible functions is that x and y coordinates switch. In fact, if we look even closer at Example 12.5's graph of $f(x)$ and $f^{-1}(x)$, we can see that $f^{-1}(x)$ is a mirror image of $f(x)$ about the line $y = x$. Let's draw the line $y = x$ on the graph in Example 12.5:



Hence, the line $y = x$ acts like a mirror, and $f(x)$ and $f^{-1}(x)$ are reflections of each other about the line $y = x$. This is no coincidence!

Recall. The definition of a composition of two functions.

Definition

Let f and g be functions of x. If f is ***composed*** of g, then

$$(f \circ g)(x) = f(g(x))$$

We say f *is composed of* g, i.e, we substitute every x in f with the function $g(x)$.

$(f \circ g)(x)$ implies that x is in the domain of $g(x)$ and $g(x)$ is in the domain of $f(x)$.

We can easily verify whether two functions are inverses of each other by using the property of the composition of $f(x)$ and $f^{-1}(x)$.

The composition of $f(x)$ and $f^{-1}(x)$

Given a function $f(x)$ to be one-to-one, and $f^{-1}(x)$ is $f(x)$'s inverse function, then

$$f\big(f^{-1}(x)\big) = f^{-1}\big(f(x)\big) = x$$

Example 12.6

Are $f(x) = \sqrt[3]{3x+4}$ and $g(x) = \dfrac{x^3-4}{3}$ inverses?

Solution.

From the property above, we can use the composition of f and g to verify whether $f\big(g(x)\big) = x$. Recall, if $f\big(g(x)\big) = x$, then f and g are inverses.

$$\begin{aligned} f\big(g(x)\big) &= f\left(\frac{x^3-4}{3}\right) \\ &= \sqrt[3]{\cancel{3}\left(\frac{x^3-4}{\cancel{3}}\right)+4} \\ &= \sqrt[3]{x^3-4+4} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

Notice, when we simplified $f(g(x))$, we obtained the simplified expression x, i.e.,

$$f(g(x)) = x$$

Thus, $f(x)$ and $g(x)$ are inverses of each other. We leave verifying $g(f(x)) = x$ to the student.

Example 12.7

Are $h(x) = 2x+5$ and $g(x) = \dfrac{x}{2} - 5$ inverses?

Solution.

We can use the composition of h and g to verify whether $h(x)$ and $g(x)$ are inverses. Recall, if $h\big(g(x)\big) = x$, then h and g are inverses.

$$\begin{aligned} h\big(g(x)\big) &= h\left(\frac{x}{2}-5\right) \\ &= 2\left(\frac{x}{2}-5\right)+5 \\ &= x-10+5 \\ &= x-5 \end{aligned}$$

Hence, $h\big(g(x)\big) \neq x$, and h and g are not inverses of each other.

12.1.3 Find the inverse of a one-to-one function algebraically

After discussing the above examples, we are able to find the inverse of a set of elements, ordered pairs, and graph, but how do we find the inverse of a one-to-one function algebraically? Luckily, we can put all the information together and obtain a method for finding a one-to-one function's inverse.

Steps for finding the inverse of a one-to-one function

Step 1. Replace the function notation with the variable y, i.e., replace $f(x)$ with y.

Step 2. Switch the independent variable and y, i.e., switch x and y variables.

Step 3. Solve for y.

Step 4. Replace y with the inverse function notation, i.e., replace y with $f^{-1}(x)$.

Step 5. Verify the composition of the original function and the obtained inverse function, i.e., $f\big(f^{-1}(x)\big) = x$ or $f^{-1}\big(f(x)\big) = x$.

Example 12.8

Find the inverse of the one-to-one function $f(x) = (x+4)^3 - 2$.

Solution.

Let's follow the steps to obtain the inverse.

Step 1. Replace $f(x)$ with y.

$$\begin{aligned} f(x) &= (x+4)^3 - 2 \\ y &= (x+4)^3 - 2 \end{aligned}$$

Step 2. Switch the x and y coordinates.

$$\begin{aligned} y &= (x+4)^3 - 2 \\ x &= (y+4)^3 - 2 \end{aligned}$$

Step 3. Solve for y.

$$\begin{aligned} x &= (y+4)^3 - 2 \\ x+2 &= (y+4)^3 \\ \sqrt[3]{x+2} &= y+4 \\ \sqrt[3]{x+2} - 4 &= y \\ y &= \sqrt[3]{x+2} - 4 \end{aligned}$$

Step 4. Replace y with $f^{-1}(x)$.

$$\begin{aligned} y &= \sqrt[3]{x+2} - 4 \\ f^{-1}(x) &= \sqrt[3]{x+2} - 4 \end{aligned}$$

Step 5. Verify the composition of $f^{-1}(x)$ and $f(x)$: $f\big(f^{-1}(x)\big) = x$ or $f^{-1}\big(f(x)\big) = x$.

$$\begin{aligned} f\big(f^{-1}(x)\big) &= f\big(\sqrt[3]{x+2} - 4\big) \\ &= \big(\sqrt[3]{x+2} - \cancel{4} + \cancel{4}\big)^3 - 2 \\ &= \big(\sqrt[3]{x+2}\big)^3 - 2 \\ &= x + \cancel{2} - \cancel{2} \\ &= x \end{aligned}$$

Since, $f\big(f^{-1}(x)\big) = x$, we verify that $f(x)$ and $f^{-1}(x)$ are, in fact, inverses.

Thus, the inverse function of $f(x)$ is $f^{-1}(x) = \sqrt[3]{x+2} - 4$.

Example 12.9

Find the inverse of the one-to-one function $g(x) = \dfrac{2x-3}{4x+2}$.

Solution.

Let's follow the steps to obtain the inverse.

Step 1. Replace $g(x)$ with y.

$$\begin{aligned} g(x) &= \frac{2x-3}{4x+2} \\ y &= \frac{2x-3}{4x+2} \end{aligned}$$

Step 2. Switch the x and y coordinates.

$$\begin{aligned} y &= \frac{2x-3}{4x+2} \\ x &= \frac{2y-3}{4y+2} \end{aligned}$$

Step 3. Solve for y.

$$\begin{aligned} x &= \frac{2y-3}{4y+2} \\ \big(4y+2\big) \cdot x &= \frac{2y-3}{\cancel{4y+2}} \cdot \cancel{\big(4y+2\big)} \\ 4xy + 2x &= 2y - 3 \end{aligned}$$

At this point, we see that there are two terms with the variable y. Hence, we should isolate the terms with y on one side, and factor y out in order to solve for y.

$$\begin{aligned} 4xy + 2x &= 2y - 3 \\ 4xy - 2y &= -2x - 3 \\ y(4x-2) &= -2x - 3 \\ y &= \frac{-2x-3}{4x-2} \end{aligned}$$

Step 4. Replace y with $g^{-1}(x)$.

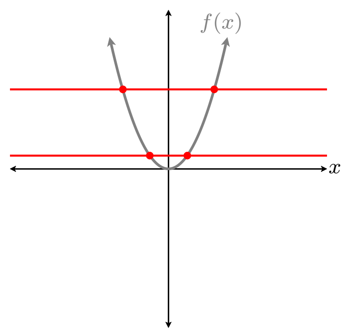
$$y = \frac{-2x-3}{4x-2}$$
$$g^{-1}(x) = \frac{-2x-3}{4x-2}$$

Step 5. Verify the composition of $g^{-1}(x)$ and $g(x)$: $g\big(g^{-1}(x)\big) = x$ or $g^{-1}\big(g(x)\big) = x$. *We leave this step for the student.*

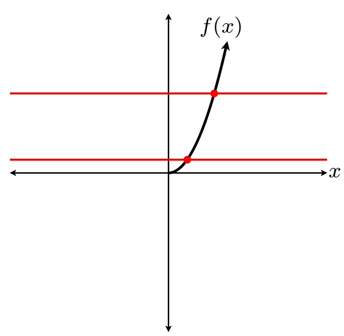
Thus, the inverse function of $g(x)$ is $g^{-1}(x) = \dfrac{-2x-3}{4x-2}$.

12.1.4 Restricting the domain of a function

In the next example, we lightly discuss the idea of *restricting the domain* of a function. See, not all functions are one-to-one, i.e., not all functions' graphs pass the *horizontal line test*. Could we make functions one-to-one? Could we force a graph of a function to pass the horizontal line test? Yes! This is where we *restrict the domain* of a function so that a piece of the function is one-to-one. Let's take a simple function from the library, $f(x) = x^2$. This function is not one-to-one:



From the figure above, the graph of $f(x) = x^2$ fails the horizontal line test. However, let's restrict the domain of $f(x)$ from $(-\infty, \infty)$ to $[0, \infty)$:



After graphing $f(x) = x^2$ on the restricted domain $[0, \infty)$, we can see $f(x)$ passes the horizontal line test, and, furthermore, is one-to-one. Careful! The function $f(x) = x^2$ is only one-to-one on the restricted domain $[0, \infty)$.

Example 12.10

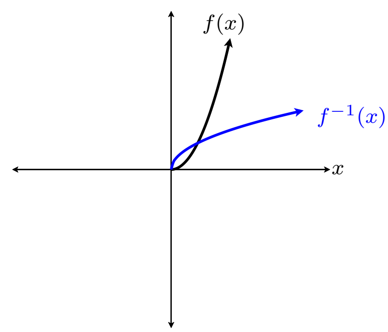
Find and graph the inverse of $f(x) = x^2$ on the restricted domain $[0, \infty)$.

Solution.

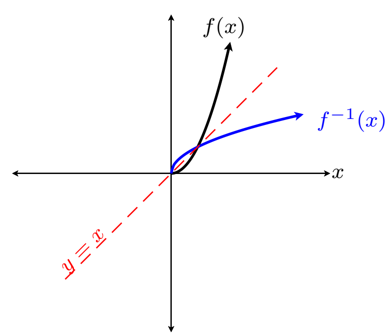
Let's first find the inverse function of $f(x)$ on the restricted domain $[0, \infty)$.

$$\begin{aligned} f(x) &= x^2 \\ y &= x^2 \\ x &= y^2 \\ \pm\sqrt{x} &= y \\ f^{-1}(x) &= \sqrt{x} \end{aligned}$$

Notice, we omit the negative value of $\sqrt{x}$ because we are on the restricted domain $[0, \infty)$, which doesn't include negative values. Thus, the positive square root is the only solution on $[0, \infty)$. Next, we can graph $f^{-1}(x) = \sqrt{x}$:



The graphs of $f(x)$ and $f^{-1}(x)$ are reflections of each other about the line $y = x$:



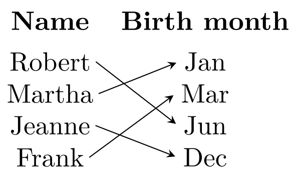
Thus, the inverse function of of $f(x) = x^2$ on the restricted domain $[0, \infty)$ is $f^{-1}(x) = \sqrt{x}$.

✎ **Note**

Restricting the domain is a useful concept that used throughout mathematics. From inverse trigonometric functions to calculating integrals of functions with vertical asymptotes. Some of these concepts we discuss early in Algebra are critical in advanced mathematics.

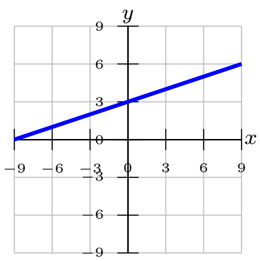
12.1.5 Inverse Functions Homework

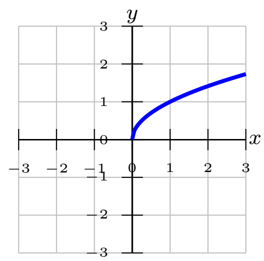
State whether the given relations are one-to-one.

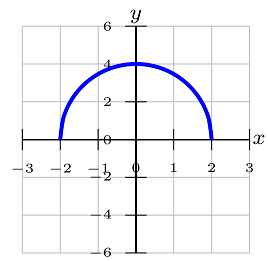
1. 

2. $\{(-2,1),(-1,-1),(0,3),(1,1),(2,3)\}$

State whether the given functions are one-to-one. If not, state a restricted domain where the function can be one-to-one.

3. 

4. 

5. 

State whether the given functions are inverses by using the composition property.

6. $g(x) = -x^5 - 3$
$f(x) = \sqrt[5]{-x-3}$

7. $f(x) = \dfrac{-x-1}{x-2}$
$g(x) = \dfrac{-2x+1}{-x-1}$

8. $g(x) = -10x + 5$
$f(x) = \dfrac{x-5}{10}$

9. $f(x) = -\dfrac{2}{x+3}$
$g(x) = \dfrac{3x+2}{x+2}$

10. $g(x) = \sqrt[5]{\dfrac{x-1}{2}}$
$f(x) = 2x^5 + 1$

11. $g(x) = \dfrac{4-x}{x}$
$f(x) = \dfrac{4}{x}$

12. $h(x) = \dfrac{-2-2x}{x}$
$f(x) = \dfrac{-2}{x+2}$

13. $f(x) = \dfrac{x-5}{10}$
$h(x) = 10x + 5$

14. $f(x) = \sqrt[5]{\dfrac{x+1}{2}}$
$g(x) = 2x^5 - 1$

15. $g(x) = \dfrac{8+9x}{2}$
$f(x) = \dfrac{5x-9}{2}$

Find the inverse function of each one-to-one function.

16. $f(x) = (x-2)^5 + 3$

17. $g(x) = \dfrac{4}{x+2}$

18. $f(x) = \dfrac{-2x-2}{x+2}$

19. $f(x) = \dfrac{10-x}{5}$

20. $g(x) = -(x-1)^3$

21. $f(x) = (x-3)^3$

22. $g(x) = \dfrac{x}{x-1}$

23. $f(x) = \dfrac{x-1}{x+1}$

24. $g(x) = \dfrac{8-5x}{4}$

25. $g(x) = -5x + 1$

26. $g(x) = -1 + x^3$

27. $h(x) = \dfrac{4-\sqrt[3]{4x}}{2}$

28. $f(x) = \dfrac{x+1}{x+2}$

29. $f(x) = \dfrac{7-3x}{x-2}$

30. $g(x) = -x$

31. $g(x) = \sqrt[3]{x+1} + 2$

32. $f(x) = \dfrac{-3}{x-3}$

33. $g(x) = \dfrac{9+x}{3}$

34. $f(x) = \dfrac{5x-15}{2}$

35. $f(x) = \dfrac{12-3x}{4}$

36. $g(x) = \sqrt[5]{\dfrac{-x+2}{2}}$

37. $f(x) = \dfrac{-3-2x}{x+3}$

38. $h(x) = \dfrac{x}{x+2}$

39. $g(x) = \dfrac{-x+2}{3}$

40. $f(x) = \dfrac{5x-5}{4}$

41. $f(x) = 3 - 2x^5$

42. $g(x) = (x-1)^3 + 2$

43. $f(x) = \dfrac{-1}{x+1}$

44. $f(x) = -\dfrac{3x}{4}$

45. $g(x) = \dfrac{-2x+1}{3}$

12.2 Exponential functions

Exponential functions take on their own set of solving and simplifying techniques since the equations are a bit different than before. For example, before we had something like $x^2 = 9$, where we could take square root of each side to solve. However, if we had something like $3^x = 9$, notice we cannot take the x^{th} root of 9 because the index is unknown. However, we may notice that $3^2 = 9$ and conclude that if $3^x = 3^2$, then $x = 2$. This is a simple example, but what if we had something a little more complex, like $10.98564^x = 34.9016$? Then the value of x isn't as obvious. These are the cases we address in this section and chapter.

World Note

One common application of exponential functions is population growth. According to the 2009 CIA World Factbook, the country with the highest population growth rate is tied between the United Arab Emirates (north of Saudi Arabia) and Burundi (central Africa) at 3.69%. There are 32 countries with negative growth rates, the lowest being the Northern Mariana Islands (north of Australia) at −7.08%.

Definition

An ***exponential function*** is a function of the form

$$f(x) = a^x,$$

where f is a function of x, $a > 0$ and $a \neq 1$.

12.2.1 Graph exponential functions

Let's start to take a look at exponential functions by looking at their graphs.

Example 12.11

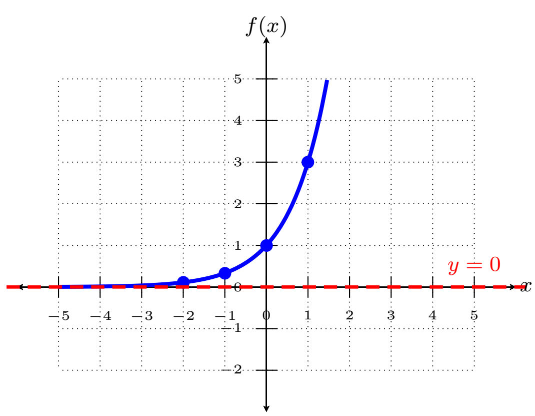
Plot $f(x) = 3^x$ by plotting points. From the graph, determine the domain of the function.

Solution.

Let's pick five x-coordinates, and find corresponding y-values. Each x-value being positive or negative, and zero. This is common practice, but not required.

x	$f(x) = 3^x$	$(x, f(x))$
-2	$f(-2) = 3^{-2} = \frac{1}{9}$	$(-2, \frac{1}{9})$
-1	$f(-1) = 3^{-1} = \frac{1}{3}$	$(-1, \frac{1}{3})$
0	$f(0) = 3^0 = 1$	$(0, 1)$
1	$f(1) = 3^1 = 3$	$(1, 3)$
2	$f(2) = 3^2 = 9$	$(2, 9)$

Plot the five ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Notice this graph is rising left to right, but, as the graph shoots to $-\infty$ (to the left), it never touches the x-axis or intersects it, resulting in a horizontal asymptote at $y = 0$. Since we see there are no restrictions to the graph, the domain is all real numbers or $(-\infty, \infty)$.



Properties of the exponential function

Property 1. The domain of an exponential function is all real numbers, i.e., $(-\infty, \infty)$.

Property 2. There are no x-intercepts; the y-intercept is at $(0, 1)$.

Property 3. If $a > 1$, then the function is an increasing function. If $0 < a < 1$, then the function is a decreasing function.

Property 4. There is a horizontal asymptote at $y = 0$, unless there is a vertical shift.

Warning!

An exponential function never crosses the x-axis. In fact, the general exponential function isn't defined at $f(x) = 0$. Take a look. If $f(x) = 0$, then $f(x) = 0 = a^x$. Ask, "For which value(s) of x such that a is raised to the power of x and the result is zero?" There exists no such x. We cannot raise a positive real number to a power and the result be zero. In the event an exponential function crosses the x-axis, then that means there was a transformation to the general exponential function.

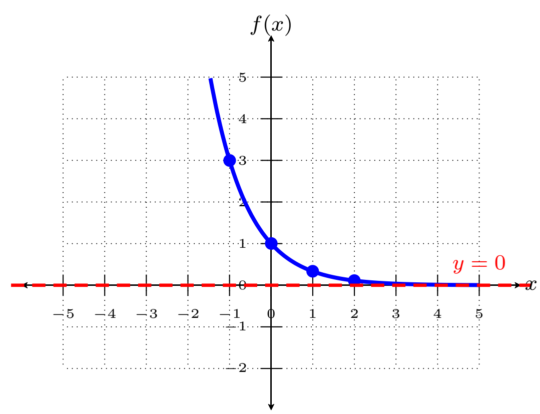
Example 12.12

Plot $f(x) = \left(\dfrac{1}{3}\right)^x$ by plotting points. From the graph, determine the domain of the function.

Solution.

Let's pick five x-coordinates, and find corresponding y-values. Each x-value being positive or negative, and zero. This is common practice, but not required.

x	$f(x) = \frac{1}{3}^x$	$(x, f(x))$
-2	$f(-2) = (\frac{1}{3})^{-2} = 9$	$(-2, 9)$
-1	$f(-1) = (\frac{1}{3})^{-1} = 3$	$(-1, 3)$
0	$f(0) = (\frac{1}{3})^{0} = 1$	$(0, 1)$
1	$f(1) = (\frac{1}{3})^{1} = \frac{1}{3}$	$(1, \frac{1}{3})$
2	$f(2) = (\frac{1}{3})^{2} = \frac{1}{9}$	$(2, \frac{1}{9})$



Plot the five ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Notice this graph is falling left to right, but, as the graph shoots to ∞ (to the right), it never touches the x-axis or intersects it. Since we see there are no restrictions to the graph, the domain is all real numbers or $(-\infty, \infty)$, and there is a horizontal asymptote at $y = 0$.

12.2.2 Exponential equations with a common base

Since the exponential function is one-to-one, we get the following.

Solving exponential equations with a common base

To solve an exponential equation with a common base on each side of the equation, we use the fact that if

$$a^m = a^n, \text{ then } m = n.$$

Example 12.13

Solve the equation: $5^{2x+1} = 125$

Solution.

We use the fact above to solve the equation.

$$\begin{aligned} 5^{2x+1} &= 125 && \text{Rewrite } 125 \text{ as } 5^3 \\ 5^{2x+1} &= 5^3 && \text{Common base, equate exponents} \\ 2x+1 &= 3 && \text{Solve for } x \\ 2x &= 2 && \text{Divide both sides by } 2 \\ x &= 1 && \text{Solution} \end{aligned}$$

We can always check the answer by verifying the solution.

$$\begin{aligned} 5^{2x+1} &\stackrel{?}{=} 125 && \text{Plug-n-chug } x = 1 \\ 5^{2(1)+1} &\stackrel{?}{=} 125 && \text{Simplify the left side} \\ 5^3 &\stackrel{?}{=} 125 && \text{Evaluate } 5^3 \\ 125 &= 125 && \checkmark \quad \text{True} \end{aligned}$$

Since we obtain a true statement by verifying the solution, then $x = 1$ is the solution.

Example 12.14

Solve the equation: $8^{3x} = 32$

Solution.

In this case, it may not seem as obvious at first, but if we rewrite each base as a common base, then we can apply the fact. Let's rewrite each base as a common base of 2.

$$\begin{aligned} 8^{3x} &= 32 && \text{Rewrite } 8 \text{ as } 2^3 \text{ and } 32 \text{ as } 2^5 \\ (2^3)^{3x} &= 2^5 && \text{Multiply exponents } 3 \text{ and } 3x \\ 2^{9x} &= 2^5 && \text{Common base, equate exponents} \\ 9x &= 5 && \text{Solve for } x \\ x &= \frac{5}{9} && \text{Solution} \end{aligned}$$

We can always verify the solution, but we leave this to the student.

Example 12.15

Solve the equation: $\left(\frac{1}{9}\right)^{2x} = 3^{7x-1}$

Solution.

In this case, it may not seem as obvious at first, but if we rewrite each base as a common base, then we can apply the fact. Let's rewrite each base as a common base of 3.

$$\begin{aligned}
\left(\frac{1}{9}\right)^{2x} &= 3^{7x-1} && \text{Rewrite } \frac{1}{9} \text{ as } \frac{1}{3^2}\\
\left(\frac{1}{3^2}\right)^{2x} &= 3^{7x-1} && \text{Rewrite } \frac{1}{3^2} \text{ as } 3^{-2}\\
(3^{-2})^{2x} &= 3^{7x-1} && \text{Multiply exponents } -2 \text{ and } 2x\\
3^{-4x} &= 3^{7x-1} && \text{Common base, equate exponents}\\
-4x &= 7x-1 && \text{Solve}\\
-11x &= -1 && \text{Isolate } x\\
x &= \frac{1}{11} && \text{Solution}
\end{aligned}$$

We can always verify the solution, but we leave this to the student.

Example 12.16

Solve the equation: $5^{4x} \cdot 5^{2x-1} = 5^{3x+11}$

Solution.

In this case, it may not seem as obvious at first, but we need to apply the product rule of exponents and obtain only one common base on each side of the equation in order to apply the fact.

$$\begin{aligned}
5^{4x} \cdot 5^{2x-1} &= 5^{3x+11} && \text{Apply product rule of exponents on the left side}\\
5^{4x+2x-1} &= 5^{3x+11} && \text{Simplify the exponent on the left side}\\
5^{6x-1} &= 5^{3x+11} && \text{Common base, equate exponents}\\
6x-1 &= 3x+11 && \text{Combine like terms}\\
3x &= 12 && \text{Isolate } x\\
x &= 4 && \text{Solution}
\end{aligned}$$

We can always verify the solution, but we leave this to the student.

Notice, the examples only present a technique for solving exponential equations with a common base. However, not all exponential equations are written with a common base. For example, something like $2 = 10^x$ cannot be written with a common base. To solve problems where we cannot rewrite the bases with a common base, we need the *logarithmic function*, which we will discuss in the next section.

12.2.3 Exponential Functions Homework

Graph each exponential function.

1. $f(x) = 4^x$

2. $x(y) = \left(\frac{1}{4}\right)^y$

3. $f(x) = -3^x$

4. $q(r) = -\left(\frac{1}{5}\right)^r$

5. $h(n) = \left(\frac{1}{2}\right)^n$

6. $g(x) = 2^x$

7. $j(x) = -2^x$

8. $k(t) = -\left(\frac{1}{2}\right)^t$

Solve the equation.

9. $3^{1-2n} = 3^{1-3n}$

10. $4^{2a} = 1$

11. $\left(\frac{1}{25}\right)^{-k} = 125^{-2k-2}$

12. $6^{2m+1} = \frac{1}{36}$

13. $6^{-3x} = 36$

14. $64^b = 2^5$

15. $\left(\frac{1}{4}\right)^x = 16$

16. $4^{3a} = 4^3$

17. $36^{3x} = 216^{2x+1}$

18. $9^{2n+3} = 243$

19. $3^{3x-2} = 3^{3x+1}$

20. $3^{-2x} = 3^3$

21. $5^{m+2} = 5^{-m}$

22. $\left(\frac{1}{36}\right)^{b-1} = 216$

23. $6^{2-2x} = 6^2$

24. $4 \cdot 2^{-3n-1} = \frac{1}{4}$

25. $4^{3k-3} \cdot 4^{2-2k} = 16^{-k}$

26. $9^{-2x} \cdot \left(\frac{1}{243}\right)^{3x} = 243^{-x}$

27. $64^{n-2} \cdot 16^{n+2} = \left(\frac{1}{4}\right)^{3n-1}$

28. $5^{-3n-3} \cdot 5^{2n} = 1$

29. $4^{2x} = \frac{1}{16}$

30. $16^{-3p} = 64^{-3p}$

31. $625^{-n-2} = \frac{1}{125}$

32. $6^{2r-3} = 6^{r-3}$

33. $5^{2n} = 5^{-n}$

34. $216^{-3v} = 36^{3v}$

35. $27^{-2n-1} = 9$

36. $4^{-3v} = 64$

37. $64^{x+2} = 16$

38. $16^{2k} = \frac{1}{64}$

39. $243^p = 27^{-3p}$

40. $4^{2n} = 4^{2-3n}$

41. $625^{2x} = 25$

42. $216^{2n} = 36$

43. $\left(\frac{1}{4}\right)^{3v-2} = 64^{1-v}$

44. $\frac{216}{6^{-2a}} = 6^{3a}$

45. $32^{2p-2} \cdot 8^p = \left(\frac{1}{2}\right)^{2p}$

46. $3^{2m} \cdot 3^{3m} = 1$

47. $3^{2-x} \cdot 3^{3x} = 1$

48. $4^{3r} \cdot 4^{-3r} = \frac{1}{64}$

12.3 Logarithmic functions

Inverse functions of exponential functions are ***logarithmic functions***, i.e., if we were to compose a logarithmic function with an exponential function (or vice versa) and the result is x ($(f \circ g)(x) = x$), then the logarithmic and exponential functions are inverses of each other. The study of logarithms is particularly interesting in many aspects of algebra, and even in advanced algebra, because they are one of the most useful functions. In this section, we introduce logarithms.

Definition

The ***logarithmic function*** is denoted by

$$y = \log_a x \text{ which is equivalent to } x = a^y,$$

where $a > 0$ and $a \neq 1$. The base is a, y is the exponent, and x is the value.

The equation $y = \log_a x$ is called the *logarithmic form* and $x = a^y$ is called the *exponential form*.

Note

When we rewrite equations in logarithmic and exponential form, we can look at the equations in a more general way so that it is obvious where we place parameters:

$$\text{exponent} = \log_{\text{base}} \text{value} \quad \textit{which is equivalent to} \quad \text{value} = \text{base}^{\text{exponent}}$$

12.3.1 Write in logarithmic and exponential form

Example 12.17

Write each exponential equation in its equivalent logarithmic form.

a) $m^3 = 5$

b) $7^2 = b$

c) $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$

Solution.

We first begin to identify the base, exponent and value. Then we rewrite the equation in logarithmic form.

a) In the equation $m^3 = 5$, we identify m is the base, 3 is the exponent, and 5 is the value.

$$3 = \log_m 5 \quad \textit{which is equivalent to} \quad 5 = m^3$$

b) In the equation $7^2 = b$, we identify 7 is the base, 2 is the exponent, and b is the value.

$$2 = \log_7 b \quad \textit{which is equivalent to} \quad b = 7^2$$

c) In the equation $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$, we identify $\frac{2}{3}$ is the base, 4 is the exponent, and $\frac{16}{81}$ is the value.

$$4 = \log_{\frac{2}{3}} \frac{16}{81} \quad \textit{which is equivalent to} \quad \frac{16}{81} = \left(\frac{2}{3}\right)^4$$

Example 12.18

Write each logarithmic equation in its equivalent exponential form.

a) $\log_x 16 = 2$ **b)** $\log_3 x = 7$ **c)** $\log_9 3 = x$

Solution.

We first begin to identify the base, exponent and value. Then we rewrite the equation in exponential form.

a) In the equation $\log_x 16 = 2$, we identify x is the base, 2 is the exponent, and 16 is the value.

$$16 = x^2 \quad \textit{which is equivalent to} \quad 2 = \log_x 16$$

b) In the equation $\log_3 x = 7$, we identify 3 is the base, 7 is the exponent, and x is the value.

$$x = 3^7 \quad \textit{which is equivalent to} \quad 7 = \log_3 x$$

c) In the equation $\log_9 3 = x$, we identify 9 is the base, x is the exponent, and 3 is the value.

$$3 = 9^x \quad \textit{which is equivalent to} \quad x = \log_9 3$$

12.3.2 Evaluate logarithmic functions

Since logarithms are just exponents, then we can use logarithms to find the exponent, or one of the other parameters, the base or value.

Example 12.19

Find the exact value: $\log_5 125$

Solution.

To find the exact value, we refrain from using any technology to obtain the answer and we only use the definition of a logarithmic function. Hence, when we see the expression $\log_5 125$, we ask,"5 to what power is 125?" because, recall, logarithms are just exponents. Some might already see the answer is 3, but let's use the definition to present a method for evaluating logarithms. Let $\log_5 125 = x$.

$$\begin{aligned} \log_5 125 &= x && \text{Rewrite in exponential form} \\ 5^x &= 125 && \text{Rewrite using common base 5} \\ 5^x &= 5^3 && \text{Common base, equate exponents} \\ x &= 3 && \text{Solution} \end{aligned}$$

We need to be careful because *we* introduced x, but x was never part of the original problem. Thus, let's write the answer properly.

$$\log_5 125 = 3$$

Example 12.20

Find the exact value: $\log_3 \frac{1}{27}$

Solution.

To find the exact value, we refrain from using any technology to obtain the answer and we only use the definition of a logarithmic function. Hence, when we see the expression $\log_3 \frac{1}{27}$, we ask,"3 to what power is $\frac{1}{27}$?" because, recall, logarithms are just exponents. Let $\log_3 \frac{1}{27} = x$.

$$\begin{aligned} \log_3 \frac{1}{27} &= x && \text{Rewrite in exponential form} \\ 3^x &= \frac{1}{27} && \text{Rewrite using common base 3} \\ 3^x &= \frac{1}{3^3} && \text{Rewrite using negative exponent } -3 \\ 3^x &= 3^{-3} && \text{Common base, equate exponents} \\ x &= -3 && \text{Solution} \end{aligned}$$

We need to be careful because *we* introduced x, but x was never part of the original problem. Thus, let's write the answer properly.

$$\log_3 \frac{1}{27} = -3$$

12.3.3 Domain of logarithmic functions

Recall. The *domain* of a function is the interval of independent values defined for that function.

Hence, it makes sense to discuss the domain of logarithmic functions. With exponential functions, the domain is all real numbers, but let's see the way it differs from the domain of a logarithmic function.

Definition

The ***domain of the logarithmic function*** is $\{x|x > 0\}$ or $(0, \infty)$, i.e., the value (or argument) of the logarithm is always positive.

Steps for obtaining the domain of logarthmic function

Given the logarithmic function $f(x) = \log_a x$, we can follow the steps below to obtain the domain.

Step 1. Identify the value of the logarithm, x. *The value will differ from x as the problems change.*

Step 2. Set the value greater than zero, i.e., $x > 0$.

Step 3. Solve the inequality as usual.

Step 4. Rewrite the inequality in interval notation, if needed.

Example 12.21

Find the domain of $f(x) = \log_5(2x + 3)$.

Solution.

We can follow the steps to obtain the domain of $f(x)$.

Step 1. The value of the given logarithm is $(2x+3)$.

Step 2. Setting the value greater than zero, we get $2x+3>0$.

Step 3. Solving the inequality as usual,

$$\begin{aligned} 2x+3 &> 0 \\ 2x &> -3 \\ x &> -\frac{3}{2} \end{aligned}$$

This means that all values for x are required to be strictly greater than $-\dfrac{3}{2}$ in order for $f(x)$ to be defined.

Step 4. Rewriting $-\dfrac{3}{2}$ in interval notation, we get $\left(-\dfrac{3}{2},\infty\right)$.

Thus, the domain of $f(x)$ is $\left\{x \middle| x>-\dfrac{3}{2}\right\}$ or, equivalently, $\left(-\dfrac{3}{2},\infty\right)$.

12.3.4 Graph logarithmic functions

Let's start to take a look at logarithmic functions by looking at their graphs. Recall, logarithmic and exponential functions are inverses of each other. Hence, we'll see that their properties also invert, i.e., x and y coordinates switch.

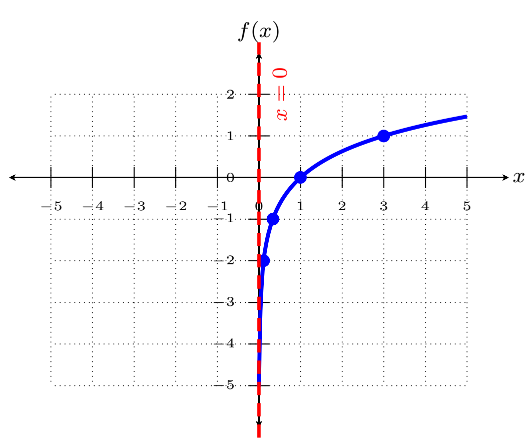
Example 12.22

Plot $f(x)=\log_3 x$ by plotting points. From the graph, determine the domain of the function.

Solution.

Let's rewrite the function as $y=\log_3 x$, and then in its equivalent exponential form: $3^y=x$. Looking at the exponential form of $f(x)$, we choose to pick y-coordinates, and find corresponding x-values. In choosing y coordinates, we can evaluate the exponential form easily.

y	$3^y=x$	$(x,f(x))$
-2	$3^{-2}=\frac{1}{9}$	$(\frac{1}{9},-2)$
-1	$3^{-1}=\frac{1}{3}$	$(\frac{1}{3},-1)$
0	$3^{0}=1$	$(1,0)$
1	$3^{1}=3$	$(3,1)$



Plot the five ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Notice this graph is rising left to right, but, as the graph shoots towards 0, it never touches the y-axis or intersects it, resulting in a vertical asymptote, $x = 0$. Since we see there is one restriction to the graph, the domain is all real numbers greater than zero or $(0, \infty)$.

Properties of the logarithmic function

Property 1. The domain of an logarithmic function is all real numbers greater than zero, i.e., $(0, \infty)$.

Property 2. There are no y-intercepts; the x-intercept is at $(1, 0)$.

Property 3. If $a > 1$, then the function is an increasing function. If $0 < a < 1$, then the function is a decreasing function.

Property 4. There is a vertical asymptote at $x = 0$, unless there is a horizontal shift.

☠ ***Warning!***

A logarithmic function never crosses the y-axis. In fact, the general logarithmic function isn't defined at $x = 0$. Take a look. If $x = 0$, then $f(0) = \log_a 0$. Ask, "a to what power is zero?" There exists no such power. We cannot raise a positive real number to a power and the result be zero. In the event a logarithmic function crosses the y-axis, then that means there was a transformation to the general logarithmic function.

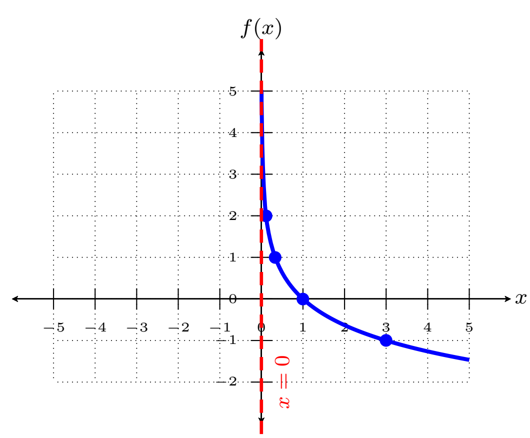
Example 12.23

Plot $f(x) = \log_{1/3} x$ by plotting points. From the graph, determine the domain of the function.

Solution.

Let's rewrite the function as $y = \log_{1/3} x$, and then in its equivalent exponential form: $\left(\frac{1}{3}\right)^y = x$. Looking at the exponential form of $f(x)$, we choose to pick y-coordinates, and find corresponding x-values. In choosing y coordinates, we can evaluate the exponential form easily.

x	$\frac{1}{3}^y = x$	$(x, f(x))$
-1	$(\frac{1}{3})^{-1} = 3$	$(3, -1)$
0	$(\frac{1}{3})^{0} = 1$	$(1, 0)$
1	$(\frac{1}{3})^{1} = \frac{1}{3}$	$(\frac{1}{3}, 1)$
2	$(\frac{1}{3})^{2} = \frac{1}{9}$	$(\frac{1}{9}, 2)$



Plot the five ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Notice this graph is falling left to right, but, as the graph shoots towards 0, it never touches the y-axis or intersects it, resulting in a horizontal asymptote at $x = 0$. Since we see there is one restriction to the graph, the domain is all real numbers greater than zero or $(0, \infty)$.

Note

This is a good time to mention two very important logarithms: the ***natural*** and ***common logarithm***.

1. The ***natural logarithm*** is given by

$$y = \log_e x = \ln x \text{ if and only if } x = e^y$$

where e is the irrational number *Euler's constant*, $e \approx 2.71828182\ldots$ Notice the $\log_e$ is replaced with ln, and that is the only difference.

2. The ***common logarithm*** is given by

$$y = \log_{10} x = \log x \text{ if and only if } x = 10^y$$

Notice the $\log_{10}$ is replaced with log, and that is the only difference. When there is no written base on the logarithm, then it is assumed it is the common logarithm (unless it is ln).

World Note

Dutch mathematician *Adriaan Vlacq* published a textbook in 1628 which listed logarithms calculated out from 1 to 100,000.

12.3.5 Solve logarithmic equations

Solving equations with logarithms has techniques that are similar when solving exponential equations. We can rewrite the logarithmic equation in its equivalent exponential form and solve.

Example 12.24

Solve for x: $\log_5 x = 2$

Solution.

We solve the equation by rewriting the equation in its equivalent exponential form and solve.

$$\begin{aligned} \log_5 x &= 2 && \text{Rewrite in exponential form} \\ 5^2 &= x && \text{Simplify} \\ 25 &= x && \text{Solution} \end{aligned}$$

Example 12.25

Solve for n: $\log_2(3n + 5) = 4$

Solution.

We solve the equation by rewriting the equation in its equivalent exponential form and solve.

$$\begin{aligned}
\log_2(3n+5) &= 4 && \text{Rewrite in exponential form} \\
2^4 &= 3n+5 && \text{Simplify } 2^4 \\
16 &= 3n+5 && \text{Isolate the variable term} \\
11 &= 3n && \text{Isolate } n \\
\frac{11}{3} &= n && \text{Solution}
\end{aligned}$$

We can see that the technique is, once the logarithm is isolated on one side of the equation, we can rewrite the equation in its equivalent exponential form and solve.

Example 12.26

Solve for t: $\log(2t-3) = -1$

Solution.

We solve the equation by rewriting the equation in its equivalent exponential form and solve. First, we see that there is no written base on the logarithm. Hence, we assume this is a common logarithm and the base is ten.

$$\begin{aligned}
\log(2t-3) &= -1 && \text{Write the common logarithm with a base 10} \\
\log_{10}(2t-3) &= -1 && \text{Rewrite in exponential form} \\
10^{-1} &= 2t-3 && \text{Simplify } 10^{-1} \\
\frac{1}{10} &= 2t-3 && \text{Isolate the variable term} \\
\frac{31}{10} &= 2t && \text{Isolate } t \\
\frac{1}{2}\cdot\frac{31}{10} &= t && \text{Simplify the left side} \\
\frac{31}{20} &= t && \text{Solution}
\end{aligned}$$

Example 12.27

Solve for a: $\ln a = 4$

Solution.

We solve the equation by rewriting the equation in its equivalent exponential form and solve. First, we see ln and assume this is a natural logarithm and the base is e.

$$\begin{aligned}
\ln a &= 4 && \text{Write the natural logarithm with a base } e \\
\log_e a &= 4 && \text{Rewrite in exponential form} \\
e^4 &= a && \text{Solution} \\
54.598 &\approx a && \text{Approximate solution for } a
\end{aligned}$$

12.3.6 Logarithmic Functions Homework

Rewrite each equation in exponential form.

1. $\log_9 81 = 2$ **2.** $\log_7 \frac{1}{49} = -2$ **3.** $\log_{13} 169 = 2$

4. $\log_b a = -16$ **5.** $\log_{16} 256 = 2$ **6.** $\log_{11} 1 = 0$

Rewrite each equation in logarithmic form.

7. $8^0 = 1$ **8.** $15^2 = 225$ **9.** $64^{1/6} = 2$

10. $17^{-2} = \frac{1}{289}$ **11.** $144^{1/2} = 12$ **12.** $19^2 = 361$

Find the exact value for each expression.

13. $\log_{125} 5$ **14.** $\log_{343} \frac{1}{7}$ **15.** $\log_4 16$ **16.** $\log_6 36$ **17.** $\log_2 64$

18. $\log_5 125$ **19.** $\log_7 1$ **20.** $\log_4 \frac{1}{64}$ **21.** $\log_{36} 6$ **22.** $\log_3 243$

Find the domain of each logarithmic function.

23. $g(x) = \log_4(7x + 10)$ **24.** $h(x) = \log_3(3x + 7)$

25. $h(x) = \log_4(7 - 8x)$ **26.** $g(x) = \log_5(4 - 8x)$

Graph each logarithmic function.

27. $f(x) = \log_4 x$ **28.** $g(x) = \log_{1/4} x$ **29.** $q(n) = -\log_2 n$

30. $h(t) = \log_{1/2} t$ **31.** $x(t) = -\log t$ **32.** $w(x) = \ln x$

Solve each equation.

33. $\log_5 x = 1$ **34.** $\log_2 x = -2$ **35.** $\log_{11} k = 2$

36. $\log_9(n + 9) = 4$ **37.** $\log_5(-3m) = 3$ **38.** $\log_{11}(x + 5) = -1$

39. $\log_4(6b + 4) = 0$ **40.** $\log_5(-10x + 4) = 4$ **41.** $\log_2(10 - 5a) = 3$

42. $\log_8 k = 3$ **43.** $\log n = 3$ **44.** $\log_4 p = 4$

45. $\log_{11}(x - 4) = -1$ **46.** $\log_2(-8r) = 1$ **47.** $\ln(-3n) = 4$

48. $\log_{11}(10v + 1) = -1$ **49.** $\log_9(7 - 6x) = -2$ **50.** $\log_8(3k - 1) = 1$

12.4 Logarithm properties

In this section, we take logarithms one step further and discuss properties of logarithms. Since logarithms are exponents, and we have many exponent properties as we learned in the *Polynomials* chapter, it makes sense we have similar properties for logarithms. E.g., if the product of two factors with the same base result in the sum of their exponents, then we have the product property of logarithms; if the quotient of two factors with the same base result in the difference of their exponents, then we have the quotient property of logarithms; a similar case for the power rule of logarithms.

12.4.1 Understand properties of logarithms

Product property of logarithms

A logarithm of a product is the sum of the logarithms:

$$\log_a(MN) = \log_a M + \log_a N$$

where a is the base, $a > 0$ and $a \neq 1$, and $M, N > 0$.

Example 12.28

Rewrite as a sum of logarithms: $\log_3(6 \cdot 5)$

Solution.

Since 3 is the base and 6 and 5 are the factors, we see in the formula $\log_a(MN)$, $a = 3$, $M = 6$, and $N = 5$. Hence,

$$\log_3(6 \cdot 5) = \log_3 6 + \log_3 5$$

Example 12.29

Rewrite as a sum of logarithms: $\ln(2k)$

Solution.

Since e is the base and 2 and k are the factors (you see this when we write $2k$ as $2 \cdot k$), we see in the formula $\log_a(MN)$, $a = e$, $M = 2$, and $N = k$. Hence,

$$\ln(2k) = \log_e(2 \cdot k) = \log_e 2 + \log_e k = \ln 2 + \ln k$$

Quotient property of logarithms

A logarithm of a quotient is the difference of the logarithms:

$$\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$$

where a is the base, $a > 0$ and $a \neq 1$, and $M, N > 0$.

Example 12.30

Rewrite as a difference of logarithms: $\log_3 \left(\frac{7}{5}\right)$

Solution.

Since 3 is the base, 7 is the numerator, and 5 is the denominator, we see in the formula $\log_a\left(\frac{M}{N}\right)$, $a = 3$, $M = 7$, and $N = 5$. Hence,

$$\log_3\left(\frac{7}{5}\right) = \log_3 7 - \log_3 5$$

Warning!

Be careful to observe that the *value of the log after the minus* sign is the *value of the denominator* of the fraction.

Example 12.31

Rewrite as a difference of logarithms: $\ln\left(\frac{7}{2}\right)$

Solution.

Since e is the base, 7 is the numerator, and 2 is the denominator, we see in the formula $\log_a\left(\frac{M}{N}\right)$, $a = e$, $M = 7$, and $N = 2$. Hence,

$$\ln\left(\frac{7}{2}\right) = \log_e\left(\frac{7}{2}\right) = \log_e 7 - \log_e 2 = \ln 7 - \ln 2$$

Power property of logarithms

A logarithm of a power is the product of the power and logarithm:

$$\log_a M^p = p\log_a M$$

where a is the base, $a > 0$ and $a \neq 1$, and $M > 0$.

Example 12.32

Rewrite all powers as factors: $\log_7 2^4$.

Solution.

Since 4 is the power on 2, then we can bring down 4 in front of the log:

$$\begin{aligned}\log_7 2^4 &= 4 \cdot \log_7 2\\ &= 4\log_7 2\end{aligned}$$

Notice 4 and $\log_7 2$ become factors.

Example 12.33

Rewrite all powers as factors: $\ln x^{\sqrt{2}}$.

Solution.

Since $\sqrt{2}$ is the power on x, then we can bring down $\sqrt{2}$ in front of the ln:

$$\begin{aligned}\ln x^{\sqrt{2}} &= \sqrt{2} \cdot \ln x \\ &= \sqrt{2} \ln x\end{aligned}$$

Notice $\sqrt{2}$ and $\ln x$ become factors.

12.4.2 Other properties of logarithms

Here are a few other properties of logarithms that we find useful when simplifying. Recall, we use these properties to have better technique when we have to solve equations with logarithms.

Other properties of logarithms

If $a, M > 0$, and $a \neq 1$, then

$$\log_a 1 = 0 \qquad \log_a a = 1$$

$$a^{\log_a M} = M \qquad \log_a a^r = r$$

Example 12.34

Evaluate each logarithm.

a) $\log_5 1$

Solution.

Since we need to find $5^? = 1$, then by the first property we know the result is zero. Thus, $\log_5 1 = 0$.

b) $\log 10$

Solution.

First, the log has no visible base. By default, we use the common logarithm and assume the base is 10. So, since we need to find $10^? = 10$, then by the second property we know the result is one. Thus, $\log 10 = 1$.

c) $\log 10^{-4}$

Solution.

First, the log has no visible base. By default, we use the common logarithm and assume the base is 10. So, since we need to find $10^? = 10^{-4}$, then by the last property we know the result is -4. Thus, $\log 10^{-4} = -4$.

d) $12^{\log_{12} \sqrt{12}}$

Solution.

If we rewrite this in logarithmic form, we get

$$\log_{12} ? = \log_{12} \sqrt{12}$$

We can easily see if this statement has to be true, then $? = \sqrt{12}$. Also, by the third property, we know the result is $\sqrt{12}$. Thus, $12^{\log_{12}\sqrt{12}} = \sqrt{12}$.

12.4.3 Expand and contract logarithms

We discuss expanding and contracting logarithmic expressions as part of applying the properties. In a future section, we apply these properties to solve logarithmic equations.

Rules of thumb for expanding logarithms

When *expanding logarithms* from a single expression, be sure to write all logarithms of

Rule 1. Products as sums

Rule 2. Quotients as differences

Rule 3. Powers as factors

We use order of operations when expanding an expression and apply the power property, and then product and quotient properties- in that order.

Example 12.35

Expand the logarithm by rewriting as a sum or difference of logarithms with powers as factors.

$$\log\left(\frac{1000\sqrt{x}}{y}\right)$$

Solution.

We see a quotient for the value of the logarithm, so we foresee we will use the quotient property of logarithms. If we look closer at the numerator, we see there is a product of two factors. Hence, we will use the product property of logarithms, too. Furthermore, we will have to use the power property of logarithms.

$$\begin{array}{rl}
\log\left(\dfrac{1000\sqrt{x}}{y}\right) & \text{Apply quotient property of logarithms} \\[2ex]
\log\left(1000\cdot\sqrt{x}\right) - \log y & \text{Apply product property of logarithms} \\
\log 1000 + \log\left(\sqrt{x}\right) - \log y & \text{Rewrite } \sqrt{x} \text{ as } x^{1/2} \\
\log 1000 + \log x^{1/2} - \log y & \text{Apply power property of logarithms} \\
\log 1000 + \dfrac{1}{2}\log x - \log y & \text{Expanded logarithmic expression}
\end{array}$$

Notice, we had to rewrite $\sqrt{x}$ as $x^{1/2}$ in order to see there was a power on x in which we had to use the product property of logarithms to bring it down as a factor. Thus, all products are written as sums, all quotients are written as differences, and all powers are written as factors.

Rules of thumb for contracting logarithms

When *contracting logarithms* from a single expression, be sure to write any

Rule 1. Multiple of a logarithm as a power of the argument

Rule 2. Sums of logarithms as a logarithm of a product

Rule 3. Differences of logarithms as a logarithm of a quotient

Example 12.36

Write $\log_2 9 + 2\log_2 x - \log_2 \left(x-4\right)$ as a single logarithm.

Solution.

Right away, we see a sum and difference with logarithms, so we know we will use the quotient and product property of logarithms. Furthermore, we will have to use the power property of logarithms.

$$\begin{aligned}
\log_2 9 + 2\log_2 x - \log_2 \left(x-4\right) &\qquad \text{Apply power property of logarithms}\\
\log_2 9 + \log_2 x^2 - \log_2 \left(x-4\right) &\qquad \text{Apply product property of logarithms}\\
\log_2 9x^2 - \log_2 \left(x-4\right) &\qquad \text{Apply quotient property of logarithms}\\
\log_2 \left(\frac{9x^2}{x-4}\right) &\qquad \text{Contracted logarithmic expression}
\end{aligned}$$

Notice, we had to rewrite $2\log_2 x$ as $\log_2 x^2$ in order to see there was a power on x in which we had to use the product property of logarithms to write 2 as the exponent. Thus, all factors are written as powers, all sums are written as products, and all differences are written as quotients.

World Note

The Scottish mathematician *John Napier* published his discovery of logarithms in 1614. His purpose was to assist in the multiplication of quantities that were then called sines. The whole sine was the value of the side of a right-angled triangle with a large hypotenuse.

12.4.4 Change of base formula

Sometimes we need to be able to rewrite logarithms in terms of other bases. This is especially helpful when counting in different numeration systems. For example, in the computer language, we count in a binary numeration system, base 2. We can use the ***change of base formula*** to rewrite numbers in different bases and it is particularly useful in computer science. However, in this textbook, we learn the *change of base formula* for the common and natural logarithm bases, i.e., base 10 and base e.

Let's take a simple general exponential equation $a^y = M$. We usually rewrite its logarithmic form as $y = \log_a M$. Well, now, let's solve for y by taking the common logarithm, log, to each side:

$$\begin{aligned}
a^y &= M &&\text{Take common logarithm to each side}\\
\log a^y &= \log M &&\text{Apply the power rule of logarithms}\\
y\log a &= \log M &&\text{Solve for } y\\
y &= \frac{\log M}{\log a} &&\text{This is the change of base formula}
\end{aligned}$$

Change of base formula

If $a, b, M > 0$, and $a, b \neq 1$, then

$$\log_a M = \frac{\log M}{\log a} \quad \text{or} \quad \log_a M = \frac{\ln M}{\ln a}$$

where log is the common logarithm, and ln is the natural logarithm. We can either formula and obtain the same result.

Example 12.37

Rewrite the expression using the Change of Base formula and then approximate the answer to three decimal places.

$$\log_2 9$$

Solution.

We would like to approximate this value using a calculator, but we cannot easily enter a logarithm in base 2. We must rewrite $\log_2 9$ so that we can easily enter it into the calculator. This is where the Change of Base (COB) formula comes in handy. Notice the base $a = 2$ and the value $M = 9$. Using the COB formula, we rewrite $\log_2 9$ as

$$\log_2 9 = \frac{\log 9}{\log 2}$$

Recall, log is the common logarithm, $\log_{10}$. Putting $\dfrac{\log 9}{\log 2}$ into the calculator, we approximate 3.170.

Note

We could have easily used the natural logarithm in the COB formula and would have obtained the same result. There's no need to use both formulas- one will suffice.

12.4.5 Logarithm Properties Homework

Write the expression as a logarithm of a single expression. Assume that variables represent positive numbers.

1. $\log_a m - \log_a n + 6\log_a k$

2. $\log_8 6 + \log_8 x$

3. $\log_8 3 + \log_8 (x^3 - 2) + \log_8 2$

4. $3\log_a(2x+1) - 2\log_a(2x-1) + 2$

Write as the sum and/or difference of logarithms. Express powers as factors.

5. $\log_4 \left(\dfrac{64}{\sqrt{x-1}}\right)$

6. $\log_2 \left(\dfrac{x^2}{y^6}\right)$

7. $\log_b (xz^3)$

8. $\log_b \left(\dfrac{xy^5}{z^7}\right)$

Use the Change of Base Formula and a calculator to evaluate the logarithm. Round to four decimal places.

9. $\log_3 23$

10. $\log_{0.4} 20$

11. $\log_{19} 57.8$

Evaluate each logarithm.

12. $\log_{23} 23$

13. $\log_{\sqrt{11}} \left(\sqrt{11}^{0.394}\right)$

14. $247^{\log_{247} \sqrt{5}}$

15. $\log_{\frac{1}{3}} 1$

12.5 Solve exponential and logarithmic equations

Even though we already discussed solving some exponential and logarithmic equations, we have yet to discuss solving exponential and logarithms equations with uncommon bases, and applying all properties. We start with a basic property of logarithms similar to the exponential equations property with common bases. Since the logarithm is one-to-one, we get the following.

12.5.1 Solve logarithmic equations using the properties of logarithms

Solving logarithmic equations with a common base

If $a, M, N > 0$, and $a \neq 1$, then

$$\log_a M = \log_a N \quad \text{implies} \quad M = N$$

Example 12.38

Solve for x: $2\log_7 x = \log_7 16$

Solution.

Since the base on both sides of the equal sign is 7, then we can rewrite the equation with $\log_7$ on each side with no coefficients in front of the logarithms.

$$\begin{aligned} 2\log_7 x &= \log_7 16 && \text{Apply the power property of logarithms} \\ \log_7 x^2 &= \log_7 16 && \text{Common base, no coefficients, equate values} \\ x^2 &= 16 && \text{Solve for } x \\ x^2 - 16 &= 0 && \text{Factor} \\ (x+4)(x-4) &= 0 && \text{Apply zero product rule} \\ x+4=0 \quad \text{or} \quad x-4 &= 0 && \text{Isolate } x \\ \cancel{x=-4} \quad \text{or} \quad x &= 4 && \text{Recall the domain of logarithms} \end{aligned}$$

Since the domain of logarithmic functions are all values greater than zero, then we eliminate $x = -4$ as a solution and have $x = 4$ as the only solution to the equation. Thus, $x = 4$ is the solution.

Example 12.39

Solve for x: $\log_4(x+6) + \log_4 x = 2$.

Solution.

We will have to use the properties of logarithms, as seen in the previous section, and the definition of a logarithm to solve this problem. There are many steps, but as long as we're organized, we will be able to obtain the solution. First, we use the product property of logarithms to rewrite the left side as a product:

$$\log_4(x+6) + \log_4 x = \log_4\big((x+6)\cdot x\big) = \log_4\big(x^2+6x\big)$$

Next, we rewrite the equation using the above and the definition of a logarithm:

$$\begin{aligned}
\log_4(x+6)+\log_4 x &= 2 && \text{Apply the product property of logarithms}\\
\log_4\left(x^2+6x\right) &= 2 && \text{Rewrite in exponential form}\\
x^2+6x &= 4^2 && \text{Simplify } 4^2\\
x^2+6x &= 16 && \text{Solve for } x\\
x^2+6x-16 &= 0 && \text{Factor}\\
(x+8)(x-2) &= 0 && \text{Apply the zero product rule}\\
x+8=0 \quad \text{or} \quad x-2 &= 0 && \text{Isolate } x\\
\cancel{x=-8} \quad \text{or} \quad x &= 2 && \text{STOP Recall the domain of logarithms}
\end{aligned}$$

Notice, $x=-8$ cannot be a solution to the equation since the value of the logarithms cannot be negative. Thus, $x=2$ is the solution to the equation.

12.5.2 Solve exponential equations

To solve exponential equations with uncommon bases, we rewrite the equations in their logarithmic form. In general, we should equate exponents when we can, but then the logarithmic form otherwise.

Example 12.40

Solve $2^x=7$. *Give the exact answer, and then use a calculator to approximate the exact answer to four decimal places.*

Solution.

When x is in the exponent, the only way to bring x down to the base position is to use the definition of a logarithm. We use this definition often when wanting to toggle between logarithmic and exponential form.

$$\begin{aligned}
2^x &= 7 && \text{Uncommon bases, rewrite in logarithmic form}\\
\log_2 7 &= x && \text{Exact answer}
\end{aligned}$$

The exact answer is $x=\log_2 7$. To approximate this value, we must use the Change of Base formula (COB):

$$\log_2 7=\frac{\log 7}{\log 2}$$

Putting this in the calculator, we get $\dfrac{\log 7}{\log 2}\approx 2.8074$. Thus, the exact answer is $x=\log_2 7$, and the approximate answer is $x=2.8074$.

Example 12.41

Solve $2e^{x+5}=5$. *Give the exact answer, and then use a calculator to approximate the exact answer to four decimal places.*

Solution.

Since we see the base of the exponential equation is e, then this is a light bulb for us to use the natural logarithmic function when using the definition of a logarithm. First, we isolate the exponential equation by dividing each side by 2, then we rewrite the statement using the definition of a logarithm.

$$\begin{aligned}
2e^{x+5} &= 5 && \text{Divide each side by a factor 2} \\
e^{x+5} &= \frac{5}{2} && \text{Uncommon bases, rewrite in logarithmic form} \\
\log_e\left(\frac{5}{2}\right) &= x+5 && \text{Rewrite } \log_e \text{ as } \ln \\
\ln\left(\frac{5}{2}\right) &= x+5 && \text{Isolate } x \\
\ln\left(\frac{5}{2}\right) - 5 &= x && \text{Exact answer}
\end{aligned}$$

Note, $x = \ln\left(\frac{5}{2}\right) - 5$ is the exact solution. To approximate this value, we put this directly in the calculator. So, we get $\ln\left(\frac{5}{2}\right) - 5 \approx -4.0837$. Thus, the exact answer is $x = \ln\left(\frac{5}{2}\right) - 5$, and the approximate answer is $x = -4.0837$.

Note

In example 12.41, we weren't required to use the COB formula since the $\boxed{\ln}$ is built directly into the scientific calculator. If the base is any number other than e, we would have to use COB prior to putting the value into the calculator. Nowadays, some calculators have a $\boxed{\log}$ button in which different bases other than 10 and e can be entered. It is just a matter of brand of calculator and identifying that feature.

We can also take the logarithm of each side of an exponential equation, as we did when developing the Change of Base Formula, to solve exponential equations.

Example 12.42

Solve $4^{7x} = 15$. *Give the exact answer, and then use a calculator to approximate the exact answer to four decimal places.*

Solution.

We can take the common logarithm of each side and solve the equation.

$$\begin{aligned}
4^{7x} &= 15 && \text{Take common logarithm of each side} \\
\log 4^{7x} &= \log 15 && \text{Apply power rule of logarithms} \\
7x\log 4 &= \log 15 && \text{Isolate } x \text{ by dividing each side by } 7\log 4 \\
x &= \frac{\log 15}{7\log 4} && \text{Exact answer}
\end{aligned}$$

Note, $x = \frac{\log 15}{7\log 4}$ is the exact solution. To approximate this value, we put this directly in the calculator. So, we get $\frac{\log 15}{7\log 4} \approx 0.2791$. Thus, the exact answer is $x = \frac{\log 15}{7\log 4}$, and the approximate answer is $x = 0.2791$.

12.5.3 Applications with exponential functions

Example 12.43

The half-life for plutonium-239 is 24,360 years. The amount A (in grams) of plutonium-239 after t years for a one-gram sample is given by $A(t) = 1 \cdot 0.5^{t/24,360}$. How long will it take before 0.6 gram of plutonium-239 is left?

Solution.

Notice the question states *how long*. Hence, we need to find time t, for a given amount A. In particularly, $A = 0.6$. Plug-n-chug $A = 0.6$ into the given function we get

$$\begin{aligned}
A(t) &= 1 \cdot 0.5^{t/24,360} && \text{Replace } A(t) = 0.6 \\
0.6 &= 1 \cdot 0.5^{t/24,360} && \text{Simplify} \\
0.6 &= 0.5^{t/24,360} && \text{Rewrite in logarithmic form} \\
\log_{0.5} 0.6 &= \frac{t}{24,360} && \text{Isolate } t \\
t &= 24,360 \cdot \log_{0.5} 0.6 && \text{Rewrite using COB}
\end{aligned}$$

$$\begin{aligned}
t &= 24,360 \cdot \frac{\log 0.6}{\log 0.5} && \text{Exact time} \\
t &\approx 17,952 && \text{Approximate time}
\end{aligned}$$

Thus, it will take about 17,952 years for plutonium-239 to reach 0.6 grams.

12.5.4 Solve Exponential and Logarithmic Equations Homework

Solve the equation.

1. $\log_5(x+2) - \log_5(x-3) = 3$

2. $\ln 60 - \ln x = \ln(x-4)$

3. $\log_8 x = \log_8 6$

4. $\log x + \log(x+1) = \log 72$

5. $\log x + \log(x-1) = \log 72$

6. $\log(3x-8) - \log 9x = 2$

7. $\log_{11}(5x-6) + \log_{11} x = 1$

Solve the equation. ***Give an exact solution and an approximate solution to four decimal places.***

8. $3^{x+7} = 7$

9. $2^{8x} = 3.6$

10. $10 \cdot 2^x = 11$

11. $\dfrac{1}{8} \cdot 5^{9x} = 4.9$

12. The half-life for thorium-227 is 18.72 days. The amount A (in grams) of thorium-239 after t years for a 10-gram sample is given by

$$A(t) = 10 \cdot 0.5^{\frac{t}{18.72}}$$

How long will it take before 4 grams of thorium-227 is left in the sample? *Round your answer to the hundredths place.*

13. According to the U.S. Census Bureau, the population of the United States in 2008 was 304 million people. In addition, the population of the United States was growing at a rate of 1.1% per year. Assuming this growth rate is continuous, the model

$$P(t) = 304 \cdot (1.011)^{t-2008}$$

represents the population P (in millions of people) in year t. According to the model, when will the population be 404 million people? *Be sure to round your answer to the nearest whole year.*

14. The formula $y = 1 + 1.5\ln(x+1)$ models the average number of free-throws a basketball player can make consecutively during practice as a function of time, where x is the number of consecutive days the basketball player has practiced for two hours. After how many days of practice can the basketball player make an average of 8 consecutive free throws?

15. Newton's Law of Cooling states that the temperature of a heated object decreases exponentially over time toward the temperature of the surrounding medium. Suppose that a coffee is served at a temperature of 143°F and placed in a room whose temperature is 70°F. The temperature μ (in °F) of the coffee at time t (in minutes) can be modeled by $\mu(t) = 70 + 73e^{-0.07t}$. When will the temperature be 105°F?

12.6 Exponential & Logarithmic Functions: Answers to the Homework Exercises

Inverse Functions

1. yes

3. yes

5. no; $[-2,0]$ or $[0,2]$

7. yes

9. no

11. no

13. yes

15. no

17. $g^{-1}(x) = \frac{4-2x}{x}$

19. $f^{-1}(x) = -5x + 10$

21. $f^{-1}(x) = \sqrt[3]{x} + 3$

23. $f^{-1}(x) = \frac{-x-1}{x-1}$

25. $g^{-1}(x) = \frac{-x+1}{5}$

27. $h^{-1}(x) = \frac{(-2x+4)^3}{4}$

29. $f^{-1}(x) = \frac{2x+7}{x+3}$

31. $g^{-1}(x) = (x-2)^3 - 1$

33. $g^{-1}(x) = 3x - 9$

35. $f^{-1}(x) = \frac{-4x+12}{3}$

37. $f^{-1}(x) = \frac{-3x-3}{x+2}$

39. $g^{-1}(x) = -3x + 2$

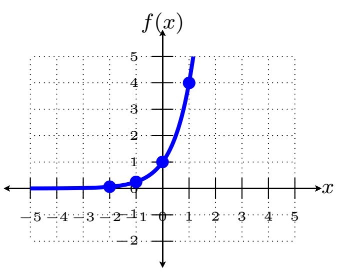
41. $f^{-1}(x) = \sqrt[5]{\frac{-x+3}{2}}$

43. $f^{-1}(x) = \frac{-1-x}{x}$

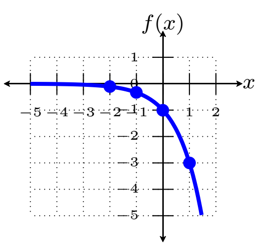
45. $g^{-1}(x) = \frac{-3x+1}{2}$

Exponential Functions

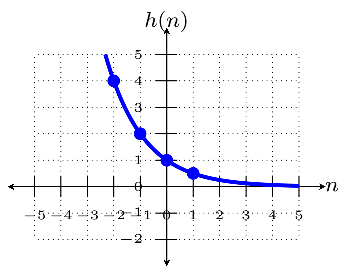
1.



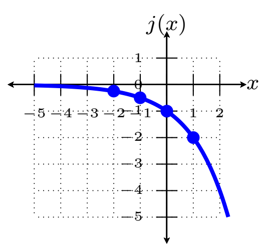
3.



5.



7.



9. 0

11. $-\frac{3}{4}$

13. $-\frac{2}{3}$

15. -2

17. No Solution

19. No Solution

21. No Solution

23. 0

25. $\frac{1}{3}$

27. $\frac{3}{8}$

29. -1

31. $-\frac{5}{4}$

33. 0

35. $-\frac{5}{6}$

37. $-\frac{4}{3}$

39. 0

41. $\frac{1}{4}$

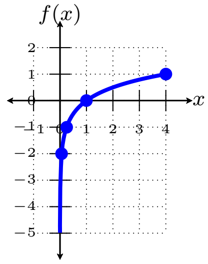
43. No Solution

45. $\frac{2}{3}$

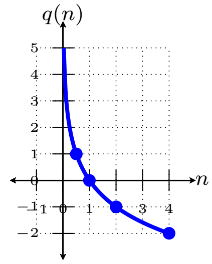
47. -1

Logarithmic Functions

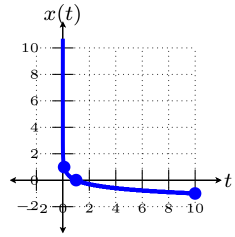
1. $9^2 = 81$
3. $13^2 = 169$
5. $16^2 = 256$
7. $\log_8 1 = 0$
9. $\log_{64} 2 = \frac{1}{6}$
11. $\log_{144} 12 = \frac{1}{2}$
13. $\frac{1}{3}$
15. 2
17. 6
19. 0
21. $\frac{1}{2}$
23. $\left(-\frac{10}{7}, \infty\right)$
25. $\left(-\infty, \frac{7}{8}\right)$
27.



29.



31.



33. 5
35. 121
37. $-\frac{125}{3}$
39. $-\frac{1}{2}$
41. $\frac{2}{5}$
43. 1,000
45. $\frac{45}{11}$
47. $-\frac{2401}{3}$
49. $\frac{283}{243}$

Logarithm Properties

1. $\log_a \left(\frac{mk^6}{n}\right)$
2. $\log_8 6x$
3. $\log_8 (6x^3 - 12)$
4. $\log_a \left(\frac{a^2(2x+1)^3}{(2x-1)^2}\right)$
5. $3 - \frac{1}{2} \log_4(x-1)$
6. $2\log_2 x - 6\log_2 y$
7. $\log_b x + 3\log_b z$
8. $\log_b x + 5\log_b y - 7\log_b z$
9. 2.8540
10. -3.2694
11. 1.3778
12. 1
13. 0.394
14. $\sqrt{5}$
15. 0

Solve Exponential and Logarithmic Functions

1. $\frac{377}{124}$
2. 10
3. 6
4. 8
5. 9
6. No Solution
7. $\frac{11}{5}$
8. $-7 + \log_3 7$; -5.2288
9. $\frac{1}{8}\log_2 3.6$; 0.2310
10. $\log_2 \frac{11}{10}$; 0.1375
11. $\frac{1}{9}\log_5 39.2$; 0.2533
12. 24.75 years
13. year 2034
14. 105 days
15. 10.5 minutes

Chapter 13

Introduction to Conics

Chapter Objectives

By the end of this chapter, the student should be able to

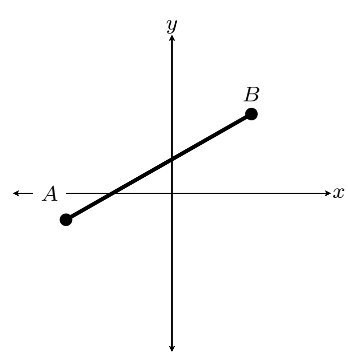
- Apply the distance and midpoint formulas to obtain the distance and midpoint, respectively, between two points
- Obtain the vertex and intercepts to graph a parabola
- Obtain the center and radius/radii to graph circles
- Rewrite nonstandard equations to the standard equations for a circle and parabola

In this chapter, we introduce conics. Even though we discussed parabolas in a previous chapter, we discuss parabolas in more of a geometric sense and use parabolas and circles in a system of equations to solve. Before we begin, let's start off with an introduction.

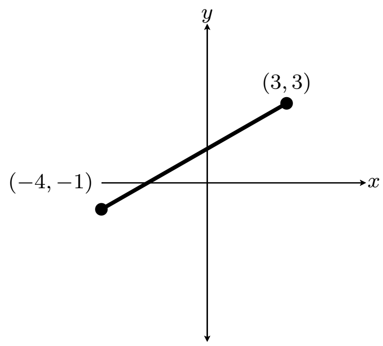
13.1 Introduction to Conics

Recall, a line segment is a line with two points on each end:

Let's take this line segment and place it on the Cartesian coordinate plane:



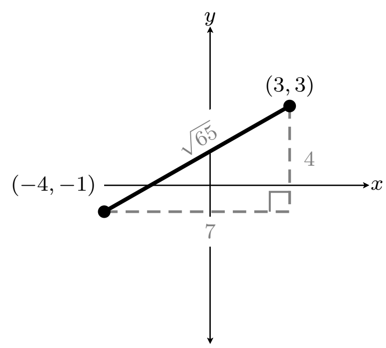
We can see that once we place the line segment on the Cartesian coordinate plane, points A and B will have x and y coordinates. Let's see the coordinates instead of A and B:



Since the line segment has endpoints in which are ordered pairs, we can find the distance of this line segment. In fact, if we used the Pythagorean Theorem to find this length, we would obtain length $\sqrt{65}$:

Pythagorean Theorem: $a^2 + b^2 = c^2$

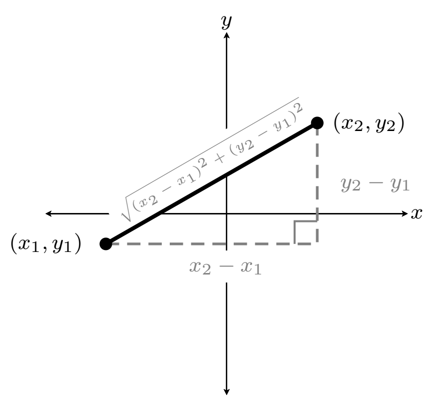
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 7^2 + 4^2 &= c^2 \\ 49 + 16 &= c^2 \\ 65 &= c^2 \\ \sqrt{65} &= c \end{aligned}$$



The next step is to find a more sophisticated way to find the distance between any two points no matter the location. Take moment to think about it. If the two points are at $(-100, 2000)$ and $(300, 5000)$, are we going to draw a right triangle that large to apply the Pythagorean Theorem? No way! Let's work smarter and not harder. Let's place two generic points where we had $(-4, -1)$ and $(3, 3)$ and apply the Pythagorean Theorem to find the hypotenuse:

Pythagorean Theorem: $a^2 + b^2 = c^2$

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 &= c^2 \\ \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= c \end{aligned}$$



Thus, we have found a generic formula to obtain the distance between any two points on the Cartesian coordinate plane.

13.1.1 The distance formula

The distance formula

Given two points, (x_1, y_1) and (x_2, y_2) on a line segment, the ***distance***, d, from (x_1, y_1) to (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 13.1

Find the distance between the points $(-2, 1)$ and $(1, 2)$. *Leave your answer in exact form, i.e., your answer should contain a square root.*

Solution.

To find the distance between the points $(-2, 1)$ and $(1, 2)$, we can apply the distance formula:

$$\begin{aligned} d &= \sqrt{(1 - (-2))^2 + (2 - 1)^2} \\ &= \sqrt{(3)^2 + (1)^2} \\ &= \sqrt{9 + 1} \\ &= \sqrt{10} \end{aligned}$$

Since the directions insisted we leave the answer in exact form, then we leave $d = \sqrt{10}$.

World Note

In the study of Euclidean geometry, we call this (most common) type of distance *Euclidean* distance, as it is derived from the Pythagorean theorem, which does not hold in non-Euclidean geometries. The *Euclidean* distance between two objects may also be generalized to the case where the objects are no longer points but are higher-dimensional manifolds, such as space curves.

13.1.2 The midpoint formula

If we can calculate the distance between two points, then we certainly should be able to find the midpoint between two points. In fact, all we do is calculate the average between the corresponding coordinates, i.e., the x coordinate of the midpoint is the average of the two given x coordinates in the ordered pairs (x_1, y_1) and (x_2, y_2):

$$x_m = \frac{x_1 + x_2}{2}$$

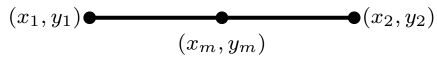
Similarly, for the y coordinate of the midpoint, it is the average of the two given y coordinates in the ordered pairs (x_1, y_1) and (x_2, y_2):

$$y_m = \frac{y_1 + y_2}{2}$$

The midpoint formula

Given two points, (x_1, y_1) and (x_2, y_2) on a line segment, the ***midpoint***, m, from (x_1, y_1) to (x_2, y_2) is given by

$$(x_m, y_m) = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$



Example 13.2

Find the midpoint of the line segment from $(-4, 2)$ to $(2, -3)$.

Solution.

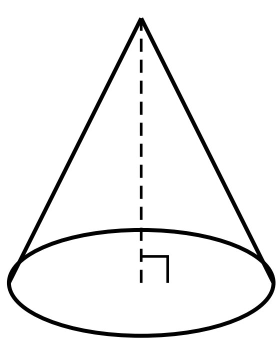
To find the midpoint between the points $(-4, 2)$ and $(2, -3)$, we can apply the midpoint formula for each coordinate:

$$\begin{aligned}(x_m, y_m) &= \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)\\ &= \left(\frac{2 + (-4)}{2}, \frac{(-3) + 2}{2}\right)\\ &= \left(\frac{-2}{2}, \frac{-1}{2}\right)\\ &= \left(-1, -\frac{1}{2}\right)\end{aligned}$$

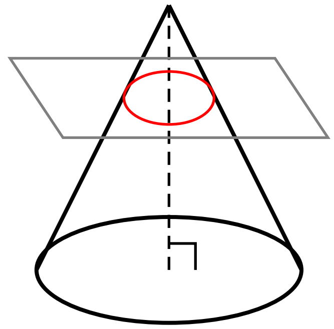
Thus, the midpoint between $(-4, 2)$ to $(2, -3)$ is $\left(-1, -\frac{1}{2}\right)$.

13.1.3 Constructing a conic

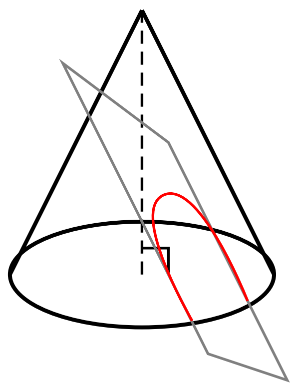
To understand the idea of conics, we begin with a shape that is familiar to all of us, a right circular cone. Let's look at a right circular cone:



Next, we can take a plane and cut through the cone so that the plane is parallel to the cone's base:



When we took the plane and cut the cone so that the plane is parallel to the cone's base, notice we made a familiar shape- the circle. Wow! So cool, right? Let's try another one. Let's take the plane and cut the cone so that the plane is parallel to the cone:

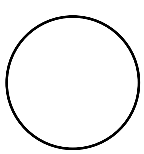


When we took the plane and cut the cone so that the plane is parallel to the cone, notice we made another familiar shape. In fact, we know this conic very well from a previous chapter- the parabola. Amazing! All we did was take a right circular cone, cut through it with a plane, and then obtained two very well-known conics.

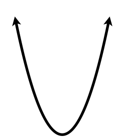
Conics

If we take a right circular cone and cut the cone so that the plane is parallel to

- the cone's base, then we obtain a ***circle***.
- the opposite side of the cone, then we obtain a ***parabola***.



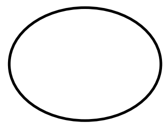
Circle



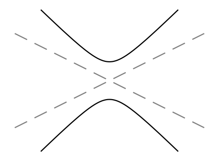
Parabola

✎ **Note**

There are two more conics, the *ellipse* and *hyperbola*, which are two other type of cuts from the cone. However, we only discuss the circle and parabola in this textbook. The ellipse and hyperbola are discussed in a future mathematics course.



Ellipse



Hyperbola

13.1.4 Introduction to Conics Homework

Find the midpoint and distance between the given two points.

1. From $(-5,-1)$ to $(5,2)$

2. From $(4,-4)$ to $(2,5)$

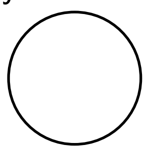
3. From $(1,1)$ to $(-2,-4)$

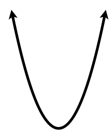
4. From $(5,-2)$ to $(-5,-5)$

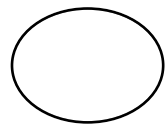
5. From $(88,-89)$ to $(97,-49)$

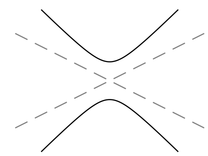
6. From $(-77,21)$ to $(-42,9)$

Identify each conic by its graph or equation.

7. 

8. 

9. 

10. 

11. $y-1=(x-3)^2$

12. $x^2+y^2=25$

13. $y-4=(x-3)^2$

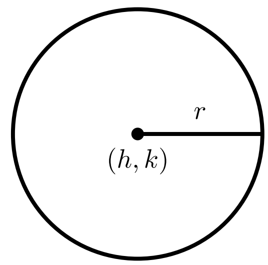
14. $x^2+y^2=16$

13.2 Circles

As we discussed in the previous section, we see a circle is simply a cut from a right circular cone. Let's discuss the properties of the circle and then graph it.

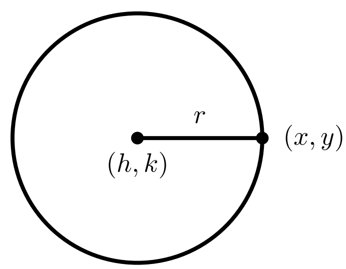
Definition

A ***circle*** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



13.2.1 The standard equation of a circle

Let's take the circle and obtain the standard equation for a circle. We take a point on the circle (x, y) and find the distance from the center to the point. We already know this distance is the radius r, but let's see algebraically.



If we apply the distance formula to find the distance between (h, k) and (x, y), we get

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

This implies that

$$r^2 = (x-h)^2 + (y-k)^2$$

Hence, by using the distance formula, we obtained the standard equation of a circle.

The standard equation of a circle

The ***standard equation of a circle*** with center (h, k) and radius $r > 0$ is

$$(x-h)^2 + (y-k)^2 = r^2$$

A circle centered at the orogin with radius r is given by

$$x^2 + y^2 = r^2$$

and we can think of h and k as horizontal and vertical shifts, respectively, as we did when graphing rational and quadratic functions.

Example 13.3

Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

Solution.

To find the standard equation of the circle with center $(-2, 3)$ and radius 5, we can apply the equation above and substitute $r = 5$, $(h, k) = (-2, 3)$.

$$
\begin{aligned}
(x-h)^2 + (y-k)^2 &= r^2 \\
(x-(-2))^2 + (y-3)^2 &= 5^2 \\
(x+2)^2 + (y-3)^2 &= 25
\end{aligned}
$$

Thus, the the standard equation of the circle with center $(-2, 3)$ and radius 5 is $(x+2)^2 + (y-3)^2 = 25$.

13.2.2 Graph a circle from the standard equation

Next, we want to graph a circle from the standard equation. Since we can easily identify the center and radius given the standard equation of a circle, we can easily graph the circle.

Example 13.4

Graph the circle $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

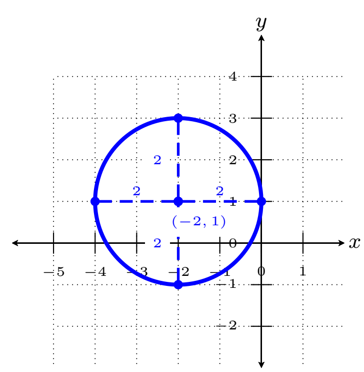
Solution.

From the standard equation of a circle, we see

$$
\begin{aligned}
(x-h)^2 + (y-k)^2 &= r^2 \\
(x+2)^2 + (y-1)^2 &= 4 \\
(x-(-2))^2 + (y-1)^2 &= 2^2
\end{aligned}
$$

From looking at the above, we see $h = -2$, $k = 1$, and $r = 2$. This implies we have a circle centered at $(-2, 1)$ with radius 2. Let's graph this information.

First, we plot the center point $(-2, 1)$. Since the distance from the center to any point on the circle is 2, then we count two units from the center in each direction. We can connect the points and construct the circle.



We can see that in order to graph a circle, we need to rewrite, if necessary, the equation of the circle in the standard equation to identify the center and radius. Then we can easily plot the center and count r units in all four directions to construct the circle.

13.2.3 Rewriting an equation of a circle in the standard equation form

What happens when we are given an equation of a circle that isn't in the form of the standard equation? We can use the method of completing the square to rewrite the equation in the standard equation form.

Recall. To complete the square of a trinomial equation of the form $x^2 + bx + c = 0$, we can add $\left(\frac{b}{2}\right)^2$ to each side of the equation.

Steps for rewriting an equation of a circle in the standard equation form

Step 1. Group the variable terms together on one side of the equation and position the constant on the opposite side of the equal sign.

Step 2. Complete the square on both variables as needed, i.e., each term should look like $(x-h)^2$ and $(y-k)^2$.

Step 3. If there are coefficients of each factor, then divide each term by the coefficient of the squares.

Example 13.5

Find the center and radius of the circle $x^2 + y^2 - 6x + 8y + 24 = 0$.

Solution.

In order to find the center and radius of the circle, we need to rewrite $x^2 + y^2 - 6x + 8y + 24 = 0$ in the form of the standard equation. Let's follow the steps to obtain the standard equation form of the given equation of the circle.

Step 1. Group the same variables together on one side of the equation and position the constant on the opposite side of the equal sign.

$$\begin{aligned} x^2 + y^2 - 6x + 8y + 24 &= 0 \\ x^2 - 6x + y^2 + 8y &= -24 \end{aligned}$$

Step 2. Complete the square on both variables as needed, i.e., each term should look like $(x-h)^2$ and $(y-k)^2$.

$$\begin{aligned} x^2 - 6x + y^2 + 8y &= -24 \\ x^2 - 6x + 9 + y^2 + 8y + 16 &= -24 + 9 + 16 \\ (x-3)^2 + (y+4)^2 &= 1 \end{aligned}$$

Step 3. Divide both sides by the coefficient of the squares.

Since the coefficients of each factor is 1, then we do not need to reduce out any coefficients.

Notice we obtained the equation $(x-3)^2 + (y+4)^2 = 1$. This is the equation $x^2 + y^2 - 6x + 8y + 24 = 0$ in its standard equation form. Hence, we can easily obtain the center and radius of the circle. From the standard equation of the circle, we see

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ (x-3)^2 + (y+4)^2 &= 1 \\ (x-3)^2 + (y-(-4))^2 &= 1^2 \end{aligned}$$

From looking at the above, we see $h = 3$, $k = -4$, and $r = 1$. This implies we have a circle centered at $(3, -4)$ with radius 1.

Example 13.6

Graph the circle in Example 13.5.

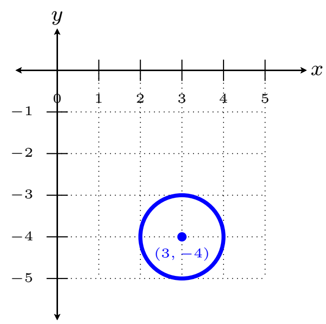
Solution.

Since we need to graph the given circle in Example 13.5, we can use the standard equation

$$(x-3)^2+(y+4)^2=1,$$

where the circle is centered at $(3,-4)$ with radius 1.

First, we plot the center point $(3,-4)$. Since the distance from the center to any point on the circle is 1, then we count one unit from the center in each direction. We can connect the points and construct the circle.



Example 13.7

Write the equation of the circle centered at $(10,7)$ that passes through $(11,-2)$.

Solution.

Since we are given the center of the circle with one point the circle passes, then we can use the standard equation to obtain the radius. Recall, the radius is the distance from the center to a point on the circle.

$$(x-h)^2+(y-k)^2=r^2$$

We can plug-n-chug $(h,k)=(10,7)$ and then $(x,y)=(11,-2)$ to find the radius.

$$\begin{aligned}(x-h)^2+(y-k)^2&=r^2\\(11-10)^2+(-2-7)^2&=r^2\\1^2+(-9)^2&=r^2\\1+81&=r^2\\82&=r^2\\\sqrt{82}&=r\end{aligned}$$

Even though the radius isn't a positive integer, it is still the radius. In fact, if we put the radius in the calculator, we would get $\sqrt{82}\approx 9.055$. Putting this all together in the standard equation, we get

$$(x-10)^2+(y-7)^2=82$$

13.2.4 Circles Homework

Write the standard equation of the circle and then sketch its graph.

1. Center $(-1,-5)$, radius 10 **2.** Center $(4,-2)$, radius 3 **3.** Center $\left(-3,\frac{7}{13}\right)$, radius $\frac{1}{2}$

Rewrite the equation into the standard equation form of a circle. Identify the center and the radius, then sketch a graph. *If it the equation is not a circle, state a reason.*

4. $x^2-4x+y^2+10y=-25$

5. $-2x^2-36x-2y^2-112=0$

6. $x^2+y^2+8x-10y-1=0$

7. $x^2+y^2+5x-y-1=0$

8. $4x^2+4y^2-24y+36=0$

9. $x^2+x+y^2-\frac{6}{5}y=1$

Find the standard equation of the circle which satisfies the given criteria.

10. center $(3,5)$, passes through $(-1,-2)$

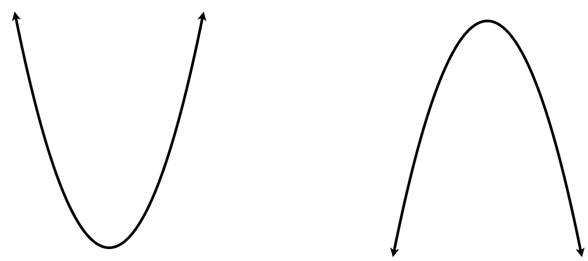
11. center $(3,6)$, passes through $(-1,4)$

12. endpoints of a diameter: $(3,6)$ and $(-1,4)$

13. endpoints of a diameter: $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$

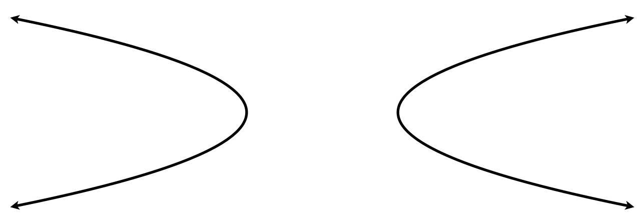
13.3 Parabolas

Recall. The graph of a quadratic function, $f(x) = ax^2 + bx + c$, is a ***parabola***.

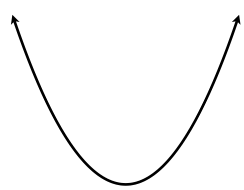


As we can see, we either had an upward or downward parabola depending on the leading coefficient, a.

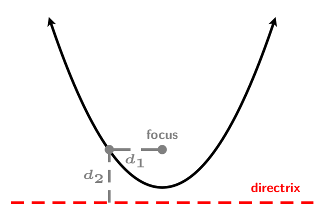
However, in this section, we look at a parabola in terms of its direction. We already discussed the parabola that opens up or down, but, now, let's look at a parabola that opens left or right:



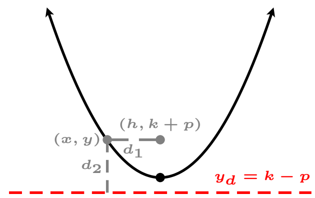
How do we obtain such parabolas? Let's start by looking at a basic upwards parabola.



There is a point above the vertex called the ***focus***. The distance from this point to the graph is the same as the distance from a line called the ***directrix*** to the same point. In the figure below, the distances d_1 and d_2 are the same, i.e., $d_1 = d_2$.



Since the focus and vertex share the same x coordinate, let's name the focus point $(h, k+p)$, where h is the x coordinate of the vertex and $k+p$ is the y coordinate of the focus. If the vertical distance from the vertex to the focus is p, then the directrix is $k - p$, where k is the y coordinate of the vertex.



Now, we can find the distance from the focus to any point on the parabola (x, y) by using the distance formula to calculate the distance between points (x, y) to $(h, k+p)$.

Example 13.8

Find the distance from a point on a parabola (x, y) to its focus $(0, p)$ with vertex at $(0, 0)$ and directrix at $y_d = -p$.

Solution.

Directly substituting the two points into the distance formula, we get

$$d = \sqrt{(x-0)^2 + (y-p)^2}$$

Recall, the distance from the focus to any point on the parabola is the same distance from the point to the directrix. Hence, the distance from the point (x, y) to the directrix is $y-(-p) = y+p$. Let's substitute this into the distance formula and rewrite the equation in a standard form for a vertical parabola:

$$\begin{aligned} y+p &= \sqrt{x^2 + (y-p)^2} \\ (y+p)^2 &= x^2 + (y-p)^2 \\ y^2 + 2py + p^2 &= x^2 + y^2 - 2yp + p^2 \\ 4yp &= x^2 \end{aligned}$$

This the resulting equation is a standard equation of a vertical parabola with vertex at $(0, 0)$.

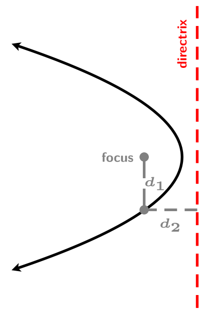
Generalizing the equation in Example 13.8 to any point on a vertical parabola with vertex (h, k), we obtain the standard equation of any vertical parabola with vertex (h, k).

The standard equation of a vertical parabola

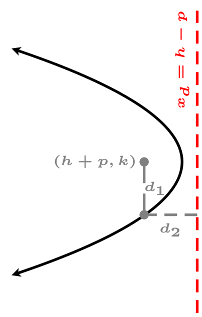
The ***standard equation of a vertical parabola*** with vertex (h, k) is

$$(x-h)^2 = 4p(y-k)$$

Continuing, if we apply the same idea to a *horizontal* parabola, where the directrix is a vertical line and the focus is to the left or right of the vertex, we get



In the figure above, the distances d_1 and d_2 are the same, i.e., $d_1 = d_2$, as it is with an upward parabola. And so, generalizing an equation for a horizontal parabola with vertex (h, k), we obtain the standard equation of any horizontal parabola with vertex (h, k).



The standard equation of a horizontal parabola

The ***standard equation of a horizontal parabola*** with vertex (h, k) is

$$(y - k)^2 = 4p(x - h)$$

13.3.1 Graph a parabola in standard equation form

In general, a parabola's direction depends on the location of the directrix and focus. Putting this all together with graphing quadratic functions, we obtain the properties of a parabola.

Properties of the parabola

To graph a parabola, we can use the table to determine the direction and all other properties.

Equation	**Direction**	**Focus**	**Directrix**	**Axis of Symmetry**
$\boldsymbol{(x-h)^2 = 4p(y-k)}$	Upward $(p > 0)$ or downward $(p < 0)$	$(h, k + p)$	$y = k - p$	$x = h$
$\boldsymbol{(y-k)^2 = 4p(x-h)}$	Left $(p < 0)$ or right $(p > 0)$	$(h + p, k)$	$x = h - p$	$y = k$

Example 13.9

Graph the parabola $(x+1)^2 = -8(y-3)$. Identify and label the vertex, focus, directrix, and axis of symmetry.

Solution.

To begin, we can identify the properties, then graph. Right away, we see the vertex is $(-1,3)$ and the axis of symmetry is $x=-1$. Since the squared factor contains x, we know this parabola opens upward or downward. Let's find p to determine the particular direction.

$$\begin{aligned}(x-h)^2 &= 4p(y-k)\\ (x+1)^2 &= -8(y-3)\end{aligned}$$

We can see from the given equation and standard equation, $4p=-8$. This implies that $p=-2$. Since $p=-2<0$, the parabola will open downward. From knowing $p=-2$, we can find the focus and directrix:

Focus:

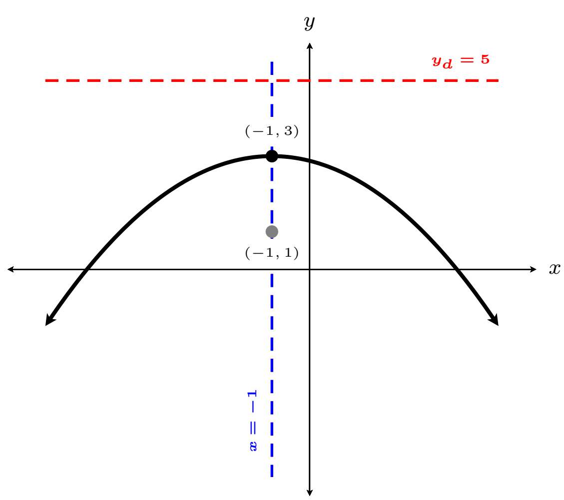
$$\begin{aligned}(h, k+p)\\ (-1, 3+(-2))\\ (-1,1)\end{aligned}$$

The focus point is $(-1,1)$.

Directrix:

$$\begin{aligned}y &= k-p\\ y &= 3-(-2)\\ y &= 5\end{aligned}$$

The directrix is the line $y=5$. We are now ready to graph the parabola with the vertex at $(-1,3)$, axis of symmetry $x=-1$, focus point at $(-1,1)$, and directrix $y=5$.



13.3.2 Rewriting an equation of a parabola in the standard equation form

Sometimes we aren't given a parabola in its standard equation form. Hence, we have to rewrite the equation so the equation of the parabola is in the standard equation. We can do this by completing the square on the term that is squared.

Recall. To complete the square of a trinomial equation of the form $x^2 + bx + c = 0$, we can add $\left(\frac{b}{2}\right)^2$ to each side of the equation.

Steps for rewriting an equation of a parabola in the standard equation form

Step 1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.

Step 2. Complete the square, if necessary, and divide by the coefficient of the perfect square, i.e., the squared term should like $(x-h)^2$ or $(y-k)^2$.

Step 3. Factor out the coefficient of the non-squared variable and the constant.

Example 13.10

Consider the equation of the parabola $y^2 + 4y + 8x = 4$. Rewrite this equation in the standard equation form and graph the parabola. Find and label the vertex, focus, directrix, and axis of symmetry.

Solution.

In order to find the properties of the parabola, we need to rewrite $y^2+4y+8x = 4$ in the form of the standard equation. Let's follow the steps to obtain the standard equation form of the given equation of the parabola.

Step 1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.

We notice the term y^2. Since the y is the squared term, then we know this parabola opens left or right.

$$\begin{aligned} y^2 + 4y + 8x &= 4 \\ y^2 + 4y &= -8x + 4 \end{aligned}$$

Step 2. Complete the square, if necessary, and divide by the coefficient of the perfect square.

$$\begin{aligned} y^2 + 4y &= -8x + 4 \\ y^2 + 4y + 4 &= -8x + 4 + 4 \\ y^2 + 4y + 4 &= -8x + 8 \\ (y+2)^2 &= -8x + 8 \end{aligned}$$

Step 3. Factor out the coefficient of the non-squared variable and the constant.

$$\begin{aligned} (y+2)^2 &= -8x + 8 \\ (y+2)^2 &= -8(x-1) \end{aligned}$$

Hence, the parabola $y^2+4y+8x=4$ in its standard equation is $(y+2)^2=-8(x-1)$. Next, let's identify all the properties to graph. We can easily see that the vertex is $(1,-2)$ and the axis of symmetry is $y=-2$. We need to find p to obtain the focus and directrix. We can see from the given equation and standard equation, $4p=-8$. This implies that $p=-2$. Since $p=-2<0$, the parabola will open to the left. From knowing $p=-2$, we can find the focus and directrix:

Focus:

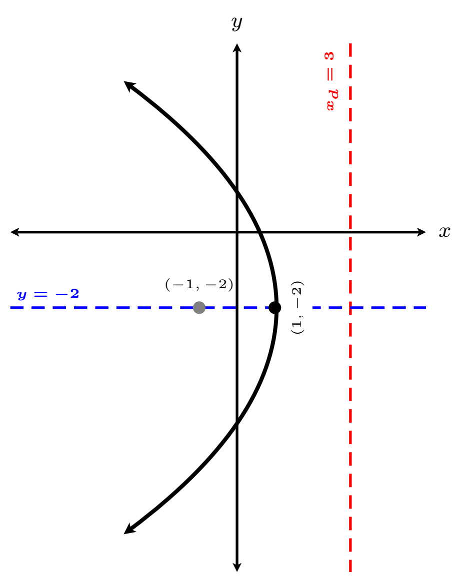
$$\begin{aligned}(h+p,k)\\(1+(-2),-2)\\(-1,-2)\end{aligned}$$

The focus point is $(-1,-2)$.

Directrix:

$$\begin{aligned}&x=h-p\\&x=1-(-2)\\&x=3\end{aligned}$$

The directrix is the line $x=3$. We are now ready to graph the parabola with the vertex at $(1,-2)$, axis of symmetry $y=-2$, focus point at $(-1,-2)$, and directrix $x=3$.



Thus, the graph above is the graph of $(y+2)^2=-8(x-1)$.

13.3.3 Parabolas Homework

Sketch the graph of each parabola. Find the vertex, focus, directrix, and axis of symmetry.

1. $(x-3)^2 = -16y$

2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$

3. $(y-2)^2 = -12(x+3)$

4. $(y+4)^2 = 4x$

5. $(x-1)^2 = 4(y+3)$

6. $(x+2)^2 = -20(y-5)$

7. $(y-4)^2 = 18(x-2)$

8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

Rewrite the equation of the parabola into its standard equation form and identify the vertex, focus, directrix, and axis of symmetry. Then sketch the graph.

9. $x^2 + 2x - 8y + 49 = 0$

10. $3y^2 - 27y + 4x + \frac{211}{4} = 0$

11. $2y^2 + 4y + x - 8 = 0$

12. $x^2 - 10x + 12y + 1 = 0$

13.4 Introduction to Conics: Answers to the Homework Exercises

Introduction to Conics

1. $\left(0, \frac{1}{2}\right)$; $\sqrt{109} \approx 10.44$

3. $\left(-\frac{1}{2}, -\frac{3}{2}\right)$; $\sqrt{34} \approx 5.83$

5. $\left(\frac{185}{2}, -69\right)$; 41

7. circle

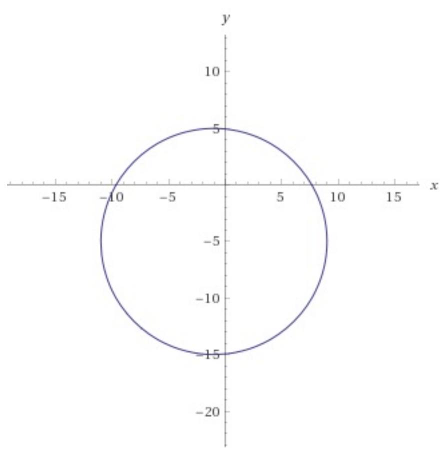
9. ellipse

11. parabola

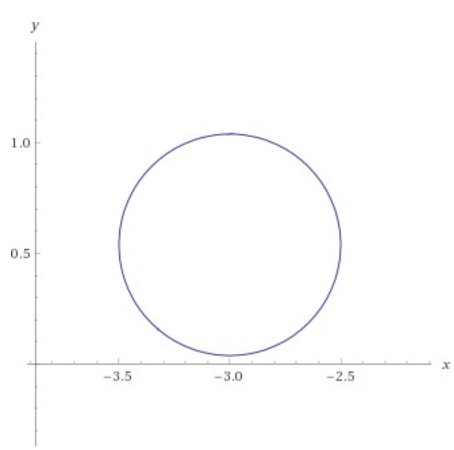
13. parabola

Circles

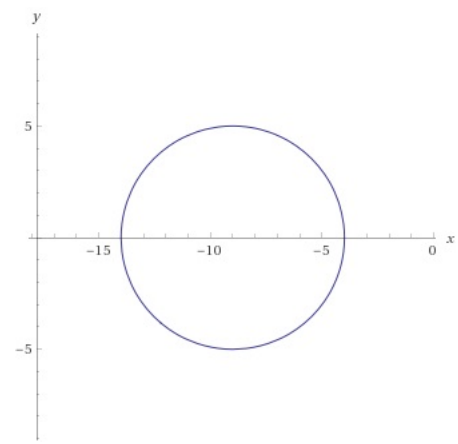
1. $(x+1)^2 + (y+5)^2 = 100$



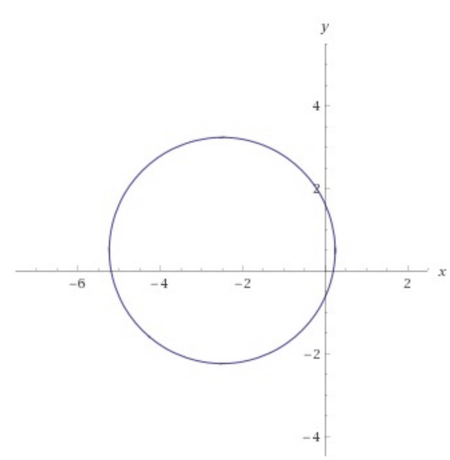
3. $(x+3)^2 + \left(y - \frac{7}{13}\right)^2 = \frac{1}{4}$



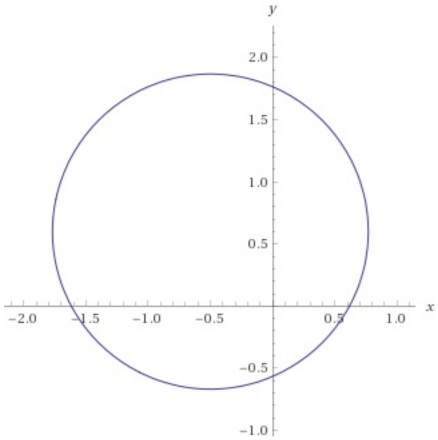
5. $(x+9)^2 + y^2 = 25$; center $(-9, 0)$, radius $r = 5$



7. $\left(x + \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{30}{4}$; center $\left(-\frac{5}{2}, \frac{1}{2}\right)$, radius $r = \frac{\sqrt{30}}{2}$



9. $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$; center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$

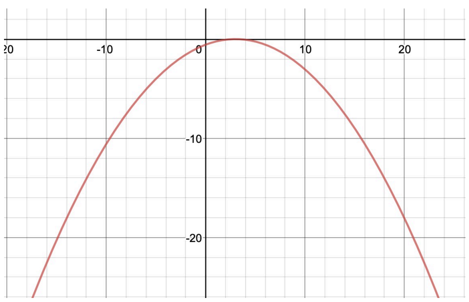


11. $(x-3)^2+(y-6)^2=20$

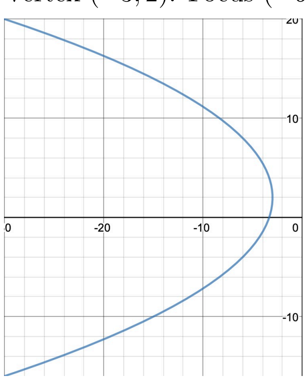
13. $(x-1)^2+\left(y-\frac{3}{2}\right)^2=\frac{13}{2}$

Parabolas

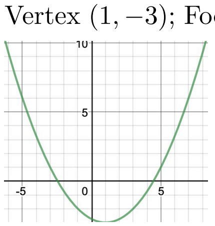
1. Vertex $(3,0)$; Focus $(3,-4)$; Directrix $y=4$



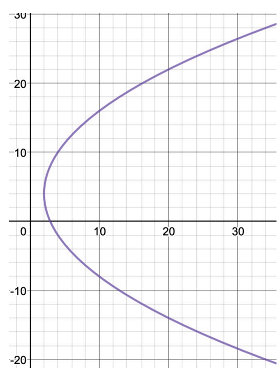
3. Vertex $(-3,2)$: Focus $(-6,2)$; Directrix $x=0$



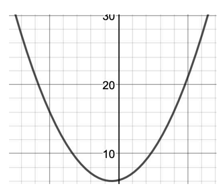
5. Vertex $(1,-3)$; Focus $(1,-2)$; Directrix $y=-4$



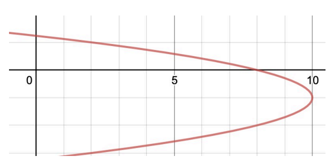
7. Vertex $(2,4)$; Focus $\left(\frac{13}{2},4\right)$; Directrix $x=-\frac{5}{2}$



9. $(x+1)^2=8(y-6)$; Vertex $(-1,6)$; Focus $(-1,8)$; Directrix $y=4$



11. $(y+1)^2=-\frac{1}{2}(x-10)$; Vertex $(10,-1)$; Focus $\left(\frac{79}{8},-1\right)$; Directrix $x=\frac{81}{8}$



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